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A BICRITERION MACHINE-ASSIGNMENT PROBLEM

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An assignment problem involving any number of machines with two objectives—one primary and another secondary—is studied. The primary objective is to minimize the duration of completing work and the secondary objective is to minimize the total processing cost. An algorithm is developed to obtain its solution.

1. INTRODUCTION

Hadley[1] has considered an assignment problem involving any number of machines with the single objective to minimize the total processing cost. Prakash[3] has considered the assignment problem involving any number of machines with the minimization of the total processing cost and the duration of completing work as primary and secondary objectives respectively. Prakash[2-4] has also considered this assignment problem with the reversed order of priorities but involving two machines only. The present paper deals with the two-objective assignment problem involving any number of machines with the minimization of the duration of completing work and the total processing cost as primary and secondary objectives respectively. A two-phase method is developed to obtain the solution of this two-objective problem. The general idea of the two-phase method is as follows : the problem is solved in two parts. First, an optimal basic feasible solution of a related assignment problem with the minimization of the total processing time and the total processing cost as primary and secondary objectives respectively is obtained. After this, the related problem is perturbed and the solution of this perturbed problem is sought. The desired solution is obtained after solving successive perturbed problems. A numerical example is given to clarify and demonstrate the solution procedure.

2. FORMULATION OF THE PROBLEM

Consider a shop with m machines of the same general type. Each machine can turn out n different products and none of the products need to be processed on more than one machine. However, their suitability for turning out different products varies because of difference in their age, size, and some other characteristics. This means that the cost and time of processing one unit of the same product will vary and depend on which machine it is processed. Let a_i be the units of time available on machine

i, b_j the number of units of product j to be processed, c_{ij} the units of cost of processing one unit of product j on machine i, t_{ij} the units of time of processing one unit of product j on machine i, and x_{ij} the number of units of product j processed on machine i in the coming time period. Let C and T denote the total processing cost and the duration of completing work respectively. It is required to determine the assignment of work to the various machines fulfilling two objectives—one primary and another secondary. The primary objective is to minimize the duration of completing work and the secondary objective is to minimize the total processing cost. The mathematical formulation of the problem is as follows : Find x_{ij}'s $\geqslant 0$ which minimize

$$T = \max \{ \sum_{j=1}^{n} t_{ij} x_{ij} : i = 1,\ldots, m\} \qquad \ldots(1)$$

and

$$C = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \qquad \ldots(2)$$

according to priorities in the order of their occurrence and subject to the constraints

$$\sum_{j=1}^{n} t_{ij} x_{ij} \leqslant a_i \; (i = 1,\ldots,m) \qquad \ldots(3)$$

$$\sum_{i=1}^{m} x_{ij} = b_j \; (j = 1,\ldots,n). \qquad \ldots(4)$$

3. Solution Procedure

The problem formulated about has two objective functions given by eqns. (1) and (2). As the objective function given by eqn. (1) is not linear, the two-objective problem is not linear. A two-phase procedure is outlined to obtain the solution of this problem. Work proceeds as follows. First, make the problem a balanced one. To do this, introduce slack variables $x_{i(n+1)} \geqslant 0$ into the constraints (3) to transform them from inequalities to equations so as to assume the form

$$\sum_{j=1}^{n} t_{ij} x_{ij} + x_i (n + 1) = a_i \quad (i = 1,\ldots,m). \qquad \ldots(5)$$

A cost zero is associated with each of the slack variables.

In the first phase of the solution procedure, a related assignment problem whose optimal basic feasible solution minimizes the total processing time

$$T' = \sum_{i=1}^{m} \sum_{j=1}^{n} t_{ij} x_{ij} \qquad \ldots(6)$$

and C given by eqn. (2) according to priorities in the order of their occurrence subject to the constraints (3) and (4) is considered. The mathematical formulation of this related assignment problem is as follows. Find x_{ij}'s $\geqslant 0$ which minimize

$$Z = M_1 \sum_{i=1}^{m} \sum_{j=1}^{n} t_{ij} x_{ij} + M_2 \sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij} x_{ij}. \quad \ldots(7)$$

Subject to the constraints (5) and (4). Here M_1 and M_2 are positive numbers subjected to the relationship that the expression $\alpha_1 M_1 + \alpha_2 M_2$ has the same sign as the nonzero α_k with the smallest subscript k in this expression. An optimal basic feasible solution for this related assignment problem is obtained in the same way as for the generalized transportation problem dealt by Hadley[1]. When Z given by eqn. (7) has attained the minimum value, the total processing time is minimized to the fullest extent possible and next the total processing cost is minimized.

Then processing times on the respective machines based on the solution of the related assignment problem are determined. With this, the second phase of the solution procedure starts. If processing times on all machines for the solution are equal, then this solution of the related assignment problem yields the desired solution. However, if processing times on all machines are not equal, then there is at least one machine on which the processing time is greater than that on at least one machine and greater than or equal to that on all other machines. Suppose that the processing time on machine s is greater than that on at least one machine and greater than or equal to that on all other machines. Then an attempt is made to reduce the processing time on machine s in such a way that the total processing cost is minimum possible while the maximum of the processing time on the respective machines is minimum. If the processing time on machine s cannot be reduced in the specified manner, then the processing time on machine s is the minimum duration of completing work and the optimal basic feasible solution of the related assignment problem yields the desired solution. If the reduced processing time on machine s is still greater than that on at least one machine and greater than or equal to that on all other machines, then the procedure is repeated. This repetition is continued till the reduced processing time on machine s is equal to that on all other machines or is less than that on some machine or is greater than that on at least one machine and greater than equal to that on all other machines but still cannot be further reduced in the specified manner. Now when the reduced processing time on machine s is either equal to that on all other machines or is greater than that on at least one machine and greater than or equal to that on all other machines but still cannot be further reduced, then the desired solution is attained. If, however, the reduced processing time on machine s is less than that on some machine, the machine on which the processing time is greater than that on at least one machine and greater than or equal to that on all other machines is selected and an attempt is made to reduce the processing time on it in the way described for that on machine s. This process is continued till the processing time on the selected machine is either equal to that on all other machines or is greater than that on at least one machine and greater than or equal to that on all other machines but still cannot be further reduced in the specified manner. And when this happens, the desired solution is attained.

A procedure to reduce the processing time on machine s in such a way that the total processing cost is minimum possible while the maximum of the processing time on the respective machines is minimum, is presented below. Let b_{n+1} denote the units of spare time available on machine s. This b_{n+1} is perturbed to the value $(b_{n+1} + \epsilon)$, where ϵ is a nonnegative number. The constraint which seeks that $(b_{n+1} + \epsilon)$ units of spare time be available on machine s after processing the requisite number of products is

$$x_s(n + 1) = b_{n+1} + \epsilon. \qquad \ldots(8)$$

Then the following perturbed problem, whose solution would yield a solution which would reduce the processing time on machine s in such a way that the total processing cost is minimum possible while the maximum of the processing time on the respective machines is minimum, is considered. Find x_{ij}'s $\geqslant 0$ which minimize Z given by eqn. (7) subject to the constraints (5), (4), (8), and at the same time assign a nonnegative value to ϵ in such a way that the total processing cost C given by eqn. (2) is minimum possible while the maximum of the processing time on the respective machines is minimum. If the assigned value of ϵ turns out to be zero, then it means that it is not possible to reduce the processing time on machine s in the specified manner and the optimal basic feasible solution of the related assignment problem yields the desired solution. However, if the assigned value of ϵ turns out to be positive, say $\epsilon = \epsilon_1$, then the spare time available on machine s increases to $(b_{n+1} + \epsilon_1)$ and the processing time on it reduces to $(a_s - b_{n+1} - \epsilon_1)$ from $(a_s - b_{n+1})$. And if it is desired to further reduce the processing time on machine s, proceed as follows : Replace b_{n+1} by $(b_{n+1} + \epsilon_1)$ in eqn. (8) which then assumes the form

$$x_{s(n+1)} = b_{n+1} + \epsilon_1 + \epsilon \qquad \ldots(9)$$

and also replace a_i by min $\{a_i, a_s - b_{n+1} - \epsilon_1\}$ in eqns. (5) which then take the form

$$\sum_{j=1}^{n} t_{ij}\, x_{ij} + x_{i(n+1)} = \min\{a_i, a_s - b_{n+1} - \epsilon_1\} \; (i = 1, \ldots, m). \qquad \ldots(10)$$

Then find x_{ij}'s $\geqslant 0$ which minimize Z given by eqn. (7) subject to the constraints (10), (4), (9), and at the same time assign a nonnegative value to ϵ in such a way that the total processing cost is minimum possible while the maximum of the processing time on the respective machines is minimum. This process of reducing maximum time on machine s is continued till the processing time on machine s is equal to that on all other machines or is less than that on some machine or is greater than that on at least one machine and greater than or equal to that on all other machines but still cannot be further reduced in the specified manner.

4. First Perturbed Problem

The first perturbed problem seeks to determine x_{ij}'s $\geqslant 0$ which minimize Z given by eqn. (7) subject to the constraints (5), (4), (8), and at the same time assign a non-negative value to ϵ in such a way that the total processing cost is minimum possible while the maximum of the processing time on the respective machines is minimum. In this perturbed problem, zero costs are associated with the slack variables as stated earlier. The procedure to solve it is indicated below.

To overcome the difficulty in obtaining an initial basic feasible solution of this perturbed problem, introduce an artificial variable $x_{(m+1)(n+1)} \geqslant 0$ into eqn. (8) which then assumes the form

$$x_{s(n+1)} + x_{(m+1)\,(n+1)} = b_{n+1} + \epsilon. \qquad \ldots(11)$$

A cost M_0 is associated with the artificial variable. Here M_0 is a positive number subjected to the relationship that the expression $\alpha_0\, M_0 + \alpha_1\, M_1 + \alpha_2\, M_2$ has the same sign as the nonzero α_k with the smallest subscript in this expression.

After this, the first perturbed problem reduces to the following equivalent problem. Find x_{ij}'s $\geqslant 0$ which minimize

$$Z' = \left\{ \begin{aligned} & M_0\, x_{(m+1)(n+1)} + M_1 \sum_{i=1}^{m} \sum_{j=1}^{n} t_{ij}\, x_{ij} \\ & \qquad + M_2 \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\, x_{ij} \end{aligned} \right\} \qquad \ldots(12)$$

subject to the constraints (5), (4), (11), and at the same time assign a nonnegative value to ϵ in such a way that the total processing cost is minimum possible while the maximum of the processing time on the respective machines is minimum. The tableau representation of this problem is shown in Table I which is identical with Table I on

TABLE I First perturbed Problem

	P_1		P_2			P_n		P_{n+1}		a_i	u_i
	$M_1 t_{11}+M_2c_{11}$		$M_1t_{12}+M_2c_{12}$			$M_1t_{1n}+M_2c_{1n}$		0			
R_1					$\ldots$						
	t_{11}	1	t_{12}	1		t_{1n}	1	1	0	a_1	u_1
	$M_1t_{21}+M_1c_{21}$		$M_1t_{22}+M_2c_{22}$			$M_1t_{2n}+M_2c_{2n}$		0			
R_2					$\ldots$					a_2	u_2
	t_{21}	1	t_{22}	1		t_{2n}	1	1	0		
	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$M_1t_{m1}+M_2c_{m1}$		$M_1t_{m2}+M_2c_{m2}$			$M_1t_{mn}+M_2c_{mn}$		0			
R_m										a_m	u_m
	t_{m1}	1	t_{m2}	1	$\ldots$	t_{mn}	1	1	0		
R_{m+1}								M_0			
			b_2						1		
b_j	b_1				$\ldots$	b_n		$b_{n+1}+\epsilon$			
v_j	v_1		v_2		$\ldots$	v_n		v_{n+1}			

page 296 of the paper by Prakash[3] if t_{ij}'s, M_0, $(M_1\, t_{ij} + M_2\, c_{ij})$'s are identified with d_{ij}'s, M, c_{ij}'s respectively. In view of this observation, it follows that the optimal basic feasible solution of the first perturbed problem can be obtained exactly in the same way as that of the problem represented by Table I of Prakash[3]. However, values of the basic variables in this solution may contain ϵ. The value of ϵ, say $\epsilon = \epsilon_1$ is so chosen that the total processing cost is minimum possible while the maximum of the processing time on the respective machines is minimum and it is ensured that none of the basic variables is negative.

After substituting the value ϵ_1 thus determined for ϵ, the optimal basic feasible solution of this perturbed problem, would yield a solution for which the spare time available on machine s is increased to $(b_{n+1} + \epsilon_1)$.

5. Numerical Example

Now the above procedure will be applied to obtain the solution of a numerical problem which results by taking $m = 3$ and $n = 5$ and assigning numerical values to all other quantities in the problem formulated in Section 2. The tableau representation of this numerical problem is shown in Table II. The left top entry of a cell (i, j)

Table II Numerical problem

	P_1	P_2	P_3	P_4	P_5	a_i
	1	1	2	2	3	
R_1						2000
	2	4	4	8	2	
	2	1	4	2	3	
R_2						3000
	1	2	3	5	4	
	1	1	1	2	2	
R_3						2400
	2	4	2	4	3	
b_j	200	500	300	100	300	

in this Table depicts the units of cost of processing one unit of product j on machine i and its left bottom entry denotes the units of time of processing one unit of product j on machine i $(i = 1, \ldots, 3;\ j = 1, \ldots, 5)$. The entries in the last column depict the units of time available on the respective machines and the entries in the last row depict the units of the respective products to be procesed.

The tableau representation of the related assignment problem associated with the numerical problem is shownin Table III. For the related assignment problem, the objective function which we seek to minimize is

$$Z = \left\{ \begin{array}{l} M_1 (2x_{11} + 4x_{12} + 4x_{13} + 8x_{14} + 2x_{15} + x_{21} \\ \quad + 2x_{22} + 3x_{23} + 5x_{24} + 4x_{25} + 2x_{31} + 4x_{32} \\ \quad + 2x_{33} + 4x_{34} + 3x_{35}) + M_2 (x_{11} + x_{12} \\ \quad + 2x_{13} + 2x_{14} + 3x_{15} + 2x_{21} + x_{22} + 4x_{23} + 2x_{24} \\ \quad + 3x_{25} + x_{31} + x_{32} + x_{33} + x_{34} + 2x_{35}) \end{array} \right\}. \quad \ldots(15)$$

To find an initial basic feasible solution for the related assignment problem, apply the column-minima method after some modification. In the case of the standard transportation problem, x_{ij} is the amount used of resource a_i; it is also the amount satisfied of requirement b_j. But in the case of the present problem, x_{ij} multiplied by the entry in the left bottom corner of the associated cell (i, j) is the amount used of resource a_i and x_{ij} multiplied by the entry in the right bottom corner of the associated cell (i, j) is the amount satisfied of requirement b_j. Following the procedure, an initial basic feasible solution is obtained, and values of the basic variables of this solution are entered in parantheses in the right top corners of the associated cells in Table III.

To determine whether the basic feasible solution is optimal, values of the relative cost coefficients corresponding to nonbasic cells are required. For this purpose, first calculate values of u_i's and v_j's and enter them in the last column and last row of Table III. Then values of the relative cost coefficients corresponding to nonbasic cells are calculated and are entered in the right top corners of the associated cells. As all the relative cost coefficients are nonnegative, the basic feasible solution is optimal. The optimal basic feasible solution of the related assignment problem yields a solution for which the total processing time is minimized to the fullest extent possible and next the total processing cost is minimized. For this solution, processing times on machines 1,2,3 are 600, 1200, 1000 units respectively; and thus the processing time on machine 2 is greater than that on all other machines. And the spare time available on machine 2 for this solution is $b_{5+1} = 1800$ units. This b_{5+1} is perturbed to the value $(b_{5+1} + \epsilon) = (1800 + \epsilon)$. The tableau representation of the perturbed problem, whose solution would reduce the processing time on machine 2 in such a way that the total processing cost is minimum possible while the maximum of the processing time on the respective machines is minimum, is shown in Table IV. The perturbed problem seeks to determine x_{ij}'s $\geqslant 0$ which minimize Z' given by

$$Z' = \left\{ \begin{array}{l} M_0 x_{46} + M_1 (2x_{11} + 4x_{12} + 4x_{13} + 8x_{14} + 2x_{15} \\ \quad + x_{21} + 2x_{22} + 3x_{23} + 5x_{24} + 4x_{25} + 2x_{31} + 4x_{32} \\ \quad + 2x_{33} + 4x_{34} + 3x_{35}) + M_2 (x_{11} + x_{12} + 2x_{13} \\ \quad + 2x_{14} + 3x_{15} + 2x_{21} + x_{22} + 4x_{23} + 2x_{24} + 3x_{25} \\ \quad + x_{31} + x_{32} + x_{33} + 2x_{34} + 2x_{35}) \end{array} \right\} \quad \ldots(16)$$

TABLE III Related assignment problem associated with numerical problem

	P_1	P_2	P_3	P_4	P_5	P_{5+1}	a_i	u_i
R_1	$2M_1+M_2$ M_1-M_2 2 1	$4M_1+M_2$ $2M_1$ 4 1	$4M_1+2M_2$ $2M_1+M_2$ 4 1	$8M_1+2M_2$ $4M_1$ 8 1	$2M_1+3M_2$ (300) 2 1	0 (1400) 1	2000	0
R_2	M_1+2M_2 (200) 1 1	$2M_1+M_2$ (500) 2 1	$3M_1+4M_2$ M_1+3M_2 3 1	$5M_1+2M_2$ M_1 5 1	$4M_1+3M_2$ $2M_1$ 4 1	0 (1800) 1	3000	0
R_3	$2M_1+M_2$ M_1-M_2 2 1	$4M_1+M_2$ $2M_1$ 4 1	$2M_1+M_2$ (300) 2 1	$4M_1+2M_2$ (100) 4 1	$3M_1+2M_2$ M_1-M_2 3 1	0 (1400) 1	2400	0
b_j	200	500	300	100	300			
v_j	M_1+2M_2	$2M_1+M_2$	$2M_1+M_2$	$4M_1+2M_2$	$2M_1+3M_2$			

TABLE IV First perturbed problem

	P_1	P_2	P_3	P_4	P_5	P_{5+1}	a_i	u_i
R_1	$2M_1+M_2$ $-M_0+M_1-M_2$ 2 1	$4M_1+M_2$ $-2M_0+2M_1$ 4 1	$4M_1+2M_2$ $2M_1+M_2$ 4 1	$8M_1+2M_2$ $4M_1$ 8 1	$2M_1+3M_2$ (300) 2 1	0 (600) 1 0	1200	0
R_2	M_1+2M_2 (200) 1 1	$2M_1+M_2$ (500) 2 1	$3M_1+4M_2$ $3M_0+M_1+2M_2$ 3 1	$5M_1+2M_2$ $5M_0+M_1$ 5 1	$4M_1+3M_2$ $4M_0+2M_1$ 4 1	0 (0) 1 1	1200	$-M_0$
R_3	$2M_1+M_2$ $-M_0+M_1-M_2$ 2 1	$4M_1+M_2$ $-2M_0+2M_1$ 4 1	$2M_1+M_2$ (300) 2 1	$4M_1+2M_2$ (100) 4 1	$3M_1+2M_2$ M_1-M_2 3 1	0 (200) 1 0	1200	0
R_{3+1}						M_0 (ϵ) 1		
b_j	200	500	300	100	300	$0+\epsilon$		
v_j	$M_0+M_1+2M_2$	$2M_0+2M_1+M_2$	$2M_1+M_2$	$4M_1+2M_2$	$2M_1+3M_2$	M_0		

and at the same time assign a nonnegative value to ϵ in such a way that the total processing cost is minimum possible while the maximum of the processing time on the respective machines is minimum.

An initial basic feasible solution for this perturbed problem is immediately obtained with the aid of the optimal solution of the related assignment problem. Values of the basic variables of this initial solution are entered in parentheses in the right top corners of the corresponding cells in Table IV. Skipping the intermediate steps, the final tableau providing the optimal basic feasible solution is shown in Table V. Values of all the relative cost coefficients appear in the right top corners of the corresponding nonbasic cells. As all the relative cost coefficients are nonnegative, the basic feasible solution is optimal. Values of the basic variables of this optimal solution are entered in parentheses in the right top corners of the corresponding cells. Processing times on the respective machines 1, 2, 3 are $(600 + 2\epsilon)$, $(1200 - \epsilon)$, 1000 units respectively while the total processing cost is $(2300 - \epsilon)$ units for this optimal solution. Now ϵ is to be assigned that nonnegative value for which the total processing cost $(2300 - \epsilon)$ is minimum possible while the maximum of the processing time on the respective machines is minimum and all x_{ij}'s are greater than or equal to zero. This means that the accepted value of ϵ is 200. After substituting $\epsilon = 200$, the optimal basic feasible solution of the perturbed problem yields a solution reducing the processing time on machine 2 to 1000 units which is equal to the processing time on the other two machines indicating that the desired solution is attained. Thus for the desired solution, the minimum duration of completing work is 1000 units and the minimum total processing cost is 2100 units.

6. Conclusions

The present work provides simple useful guidelines for an optimal assignment of work to machines in a shop. The only input data needed to use the model considered in this work are the units of time available on different machines, the number of units of different products to be processed, costs and times of processing one unit of each of the products on each of the machines.

The model considered in this work is quite versatile because a variety of real-life constraints other than those already included in it can be easily accommodated within its frame work. For instance, the additional condition that a specified machine cannot be used due to being out of order can be met easily by taking the cost and time of processing one unit of each of the products on it to be infinite.

The present work has applications in areas other than a shop. This can be seen if the term "machine" is stretched to include a production unit of a company. With this generalization, the model considered in this work is applicable to the situation when a company seeks to determine an optimal assignment of work among its production units located at different places each producing the same type of different

TABLE V Final tableau providing optimal basic feasible solution

	P_1		P_2		P_3		P_4		P_5		P_{5+1}		a_i	u_i
R_1	$2M_1+M_2$	(ϵ)	$4M_1+M_2$	$2M_2$	$4M_1+M_2$	$12M_1+2M_2$	$8M_1+2M_2$	$4M_1$	$2M_1+3M_2$	(300)	0	$(600-2\epsilon)$	1200	0
	2	1	4		4	1	8	1	2	1	1	0		
R_2	M_1+2M_2	$(200-\epsilon)$	$2M_1+M_2$	(500)	$3M_1+4M_2$	$4M_1$	$5M_1+2M_2$	$6M_1-5M_2$	$4M_1+3M_2$	$6M_1-4M_2$	0	(ϵ)	1200	$-M_1+M_2$
	1	1	2	1	3	1	5	1	4	1	1	1		
R_3	$2M_1+M_2$	0	$4M_1+M_2$	$2M_2$	$2M_1+M_2$	(300)	$4M_1+2M_2$	(100)	$3M_1+2M_2$	M_1-M_2	0	(200)	1200	0
	2	1	4	1	2	1	4	1	3	1	1	0		
R_{3+1}											M_0			
												1		
b_j	200		500		300		100		300		$0+\epsilon$			
v_j	$2M_1+M_2$		$4M_1-M_2$		$2M_1+M_2$		$4M_1+2M_2$		$2M_1+3M_2$		M_1-M_2			

products but the cost and time of production of one unit of the same product varies and depends on the production unit at which it is produced due to variations in the costs of labour and raw materials in the vicinity of the production unit and also due to other reasons.

Future extension of the present work is possible. The two-objective problem dealt in this work may be considered without assigning priorities to the two objectives and the set of non-dominated solutions may be generated.

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