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ON A CLASS OF ANALYTIC FUNCTIONS I

RAJALAKSHMI RAJAGOPAL

Department of Mathematics, Loyola College, Madras

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Let f, g, h be analytic in $E = \{z : |z| < 1\}$ satisfying $f(0) = g(0) = h(0) = 0$ and $f'(0) = g'(0) = h'(0) = 1$. Two classes $R(\alpha)$ and $U(\alpha)$ are defined as follows. $R(\alpha')$, $\alpha > 0$ consists of functions f, with $f(z) \neq 0$ $(0 < |z| < 1)$ and $f'(z) \neq 0$ such that

$$\text{Re}\{z^{\alpha+1} f'^{\alpha}(z) f^{1-\alpha}(z)/g(z) h(z)\} > 0, z \in E,$$

where g and h are subjected to the following conditions (i) $g \in S^*$, $h \in K$ (ii) $g \in K$, $h \in K$, S^* and K being the classes of starlike and convex functions, respectively. $U(\alpha)$, $\alpha > 0$ consists of functions f, with $f(z) \neq 0$ $(0 < |z| < 1)$ and $f'(z) \neq 0$ such that

$$\text{Re}\{z^{\alpha-1} f'^{\alpha}(z) f^{1-\alpha}(z)/g'(z)\} > 0, z \in E,$$

where g is an odd function, convex in the direction of imaginary axis.
In this paper some properties of these classes are investigated.

1. INTRODUCTION

Let S^* be the class of functions $g(z) = z + g_2 z^2 + \ldots$ which are regular, univalent and star-like in $E = \{z : |z| < 1\}$.

Let K be the class of functions $h(z) = z + h_2 z^2 + \ldots$ which are regular, univalent and convex in E.

Let $M(\alpha)$, α real, denote the class of functions f analytic in E, with $f(0) = 0$, $f'(0) = 1$, $f(z) \neq 0$ $(0 < |z| < 1)$ and $f'(z) \neq 0$ in E satisfying the condition

$$\text{Re}\left\{\alpha\left(1 + \frac{zf''(z)}{f'z}\right) + (1 - \alpha)\frac{zf'(z)}{f(z)}\right\} > 0, z \in E.$$

Functions belonging to this class are called α-convex[4].

In this paper we introduce two classes $R(\alpha)$ and $U(\alpha)$ and investigate certain properties of functions belonging to these classes.

Let $R(\alpha)$, $\alpha > 0$, denote the class of functions with $f(0) = 0$, $f'(0) = 1$, $f(z) \neq 0$ $(0 < |z| < 1)$ and $f'(z) \neq 0$, $z \in E$ satisfying the condition.

$$\text{Re}\frac{z^{\alpha+1} f'^{\alpha}(z) f^{1-\alpha}(z)}{g(z) h(z)} > 0, z \in E$$

where g and h are subjected to the following conditions (i) $g \in S^*$ $h \in K$ (ii) $g \in K$, $h \in K$. If we put $\alpha = 1$, $R(\alpha)$ reduces to the class introduced by Bharati[1]. Taking $h(z) = z$ we obtain the class of α-close-to-convex functions defined by Bharati[2].

Let $U(\alpha)$, $\alpha > 0$, denote the class of function f with $f(0) = 0, f'(0) = 1$, $f(z) \neq 0$ $(0 < |z| < 1)$ and $f'(z) \neq 0$ satisfying in E the condition

$$\operatorname{Re} \frac{z^{\alpha-1} z^{\alpha-1} f'^{\alpha}(z) f^{1-\alpha}(z)}{g'(z)} > 0, z \in E$$

where g is an odd function convex in the direction of imaginary axis. For $\alpha = 1$ this reduces to the class introduced by Gupta[3].

2. Radius of α-Convexity

Theorem 1—Let $f \in R(\alpha)$, $\alpha > 0$. Let $g \in S^*$ and $h \in K$. Then f is α-convex in $|z| < \frac{5-\sqrt{17}}{4}$. The bound is sharp.

Proof : If we set $z^{\alpha+1} f'^{\alpha}(z) f^{1-\alpha}(z)/g(z) h(z) = p(z)$ then $\operatorname{Re} p(z) > 0$ for $z \in E$.

Logarithmic differentiation yields

$$\alpha \left(\frac{1+zf''(z)}{f'(z)} \right) + (1-\alpha) \frac{zf'(z)}{f(z)} = \frac{zp'(z)}{p(z)} + \frac{zh'(z)}{h(z)} + \frac{zg'(z)}{g(z)} - 1. \tag{2.1}$$

We use the following well known estimates for starlike functions, convex functions and functions with positive real part.

$$\operatorname{Re} \frac{zg'(z)}{g(z)} \geqslant \frac{1+|z|}{1-|z|} \quad z \in E, g \in S^*$$

$$\operatorname{Re} \frac{zh'(z)}{h(z)} \geqslant \frac{1}{1+|z|} \quad z \in E, h \in K$$

$$\operatorname{Re} \frac{zp'(z)}{p(z)} \geqslant \frac{-2|z|}{1-|z|^2} \quad z \in E, \operatorname{Re} p(z) > 0.$$

Then from (2.1)

$$\operatorname{Re} \left\{ \alpha \left(1 + \frac{zf''(z)}{f'(z)} \right) + (1-\alpha) \frac{zf'(z)}{f(z)} \right\} \geqslant \frac{1-5r+2r^2}{1-r^2}.$$

Hence f is α-convex when $|z| = r < \frac{5-\sqrt{17}}{4}$. By taking $g(z) = \frac{z}{1+z}$, $h(z) = \frac{z}{1+z}$ and $p(z) = \frac{1-z}{1+z}$, we see that the estimate obtained above is sharp.

Theorem 2—Let $f \in R(\alpha)$, $\alpha > 0$. Let $g \in K$ and $h \in K$. Then f is α-convex in $|z| < 2 - \sqrt{3}$. The result is sharp.

The proof of this theorem is similar to that of Theorem 1.

Putting $\alpha = 1$ in Theorems 1 and 2, we deduce the corresponding results of Bharati[1].

We use the following lemma to prove the next theorem.

Lemma[3]—Let g be an odd function which is convex in the direction of imaginary axis. Then

$$\operatorname{Re} \frac{zg''(z)}{g(z)} \geqslant \frac{-6|z|^2 + 2|z|^4}{1 - |z|^4}, \quad z \in E.$$

Theorem 3—Let $f \in U(\alpha)$, $\alpha > 0$. Then f is α-convex in $|z| < 2 - \sqrt{3}$.

PROOF : By hypothesis,

$$z^{\alpha-1} f'^{\alpha}(z) f^{1-\alpha}(z)/g'(z) = p(z)$$

where $\operatorname{Re} p(z) > 0$, $z \in E$.

Logarithmic differentiation yields

$$\alpha \left(1 + \frac{zf''(z)}{f'(z)}\right) + (1 - \alpha) \frac{zf'(z)}{f(z)} = \frac{zp'(z)}{p(z)} + \frac{zg''(z)}{g'(z)} + 1. \quad ...(2.2)$$

From (2.2), by using the above lemma, we get

$$\operatorname{Re} \left\{\alpha \left(1 + \frac{zf''(z)}{f'(z)}\right) + (1 - \alpha) \frac{zf(z)}{f(z)}\right\} \geqslant \frac{(1 + r)(r^2 - 4r + 1)}{(1 - r)(1 + r^2)}.$$

Hence f is α-convex, in $|z| = r < 2 - \sqrt{3}$. Setting $\alpha = 1$ we deduce the result of Gupta[3].

3. COEFFICIENT ESTIMATES

Theorem 4—Let $f \in R(\alpha)$, $\alpha > 0$, $g \in S^*$ and $h \in K$. If $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$ then

$$|a_2| \leqslant \frac{5}{1 + \alpha}, \quad \alpha > 0$$

$$|a_3| \leqslant \frac{3\alpha^2 + 81\alpha + 28}{2(1 + \alpha)^2 (1 + 2\alpha)}, \quad 0 < \alpha \leqslant 1.$$

The bounds are sharp.

PROOF : Let

$$z^{\alpha+1} f'^{\alpha}(z) f^{1-\alpha}(z) = p(z) g(z) h(z) \quad ...(3.1)$$

where $\operatorname{Re} p(z) > 0$, $z \in E$.

From (3.1) we get

$$a_2 (1 + \alpha) = g_2 + h_2 + p_1 \qquad ...(3.2)$$

$$a_3 (1 + 2\alpha) - \frac{\alpha (1 - \alpha)}{2} a_2^2 = h_3 + g_3 + h_2 g_2 + p_1 (g_2 + h_2) + p_2. \qquad ...(3.3)$$

We use the following well-known coefficient estimates

$$| g_n | \leqslant n, | h_n | \leqslant 1 \text{ and } | p_n \leqslant 2, n = 1, 2, 3... \qquad ...(3.4)$$

Using (3.4) in (3.2) and (3.3) we obtain the required estimates. Taking $g(z) = \frac{z}{(1+z)^2}$; $h(z) = \frac{z}{1+z}$ and $p(z) = \frac{1-z}{1+z}$. We find that the function defined by $z^{\alpha-1} f'^{\alpha}(z)$ $f^{1-\alpha}(z) = \frac{1-z}{(1+z)^3}$ serves as an extremal function in the above theorem.

Theorem 5—Let $f \in R(\alpha)$, $\alpha > 0$ $g \in K$ and $h \in K$. If $f(z) = z + a_2 z^2 + a_3 z^3 + ...$ then

$$| a_2 | \leqslant \frac{4}{1 + \alpha}, \alpha > 0$$

$$| a_3 | \leqslant \frac{\alpha^2 + 26\alpha + 9}{(1 + \alpha)^2 (1 + 2\alpha)}, 0 < \alpha \leqslant 1.$$

The bounds are sharp.

The proof of this theorem is similar to that of Theorem 4. Setting $\alpha = 1$ in Theorems 4 and 5 we deduce the results obtained by Bharati[1].

Theorem 6—Let $f \in U(\alpha)$. $\alpha > 0$. Suppose that g is an odd function convex in the direction of imaginary axis.

If $f(z) = z + a_2 z^2 + ...$ then

$$| a_2 | \leqslant \frac{2}{1 + \alpha}, \alpha > 0$$

$$| a_3 | \leqslant \frac{3\alpha^2 + 12\alpha + 5}{(2\alpha + 1)(1 + \alpha)^2}, 0 < \alpha \leqslant 1.$$

PROOF : As $f \in U(\alpha)$ we have

$$z^{\alpha-1} f'^{\alpha}(z) f^{1-\alpha}(z) = p(z) g'(z)$$

where Re $p(z) > 0$, $z \in E$.

Taking

$$g(z) = z + g_3 z^3 + g_5 z^5 + ... \qquad ...(3.9)$$

we obtain

$$a_2 (1 + \alpha) = p_1 \ldots \quad \ldots(3.10)$$

$$a_3 (2\alpha + 1) + \alpha (1 - \alpha) \frac{a_2^2}{2} = p_2 + 3g_3 \ldots \quad \ldots(3.11)$$

Using the well known estimates $| p_n | \leqslant 2$, $| g_{n+1} | \leqslant 1$, $n = 1, 2, \ldots$ in (3.10) and (3.11) we get the required estimates.

Putting $\alpha = 1$, we deduce the result of Gupta[3].

4. Integral Representation

Theorem 7—A function $f \in R(\alpha)$ if and only if there exists a starlike function g, a convex function h and an analytic function p having positive real part in E such that

$$f(z) = \left\{ \frac{1}{\alpha} \int_0^z t^{-1-1/\alpha} p^{1/\alpha}(t) g^{1/\alpha}(t) h^{1/\alpha}(t) dt \right\}^{\alpha} \quad \alpha > 0.$$

Proof : Since $f \in R(\alpha)$

$$z^{\alpha+1} f'^{\alpha}(z) f^{1-\alpha}(z) / g(z) h(z) = p(z)$$

$$\text{Re } p(z) > 0, z \in E.$$

From this it follows that

$$f'(z) f^{1/\alpha - 1}(z) = p^{1/\alpha}(z) g^{1/\alpha}(z) h^{1/\alpha}(z) z^{-1-1/\alpha}.$$

Multiplying both sides by $1/\alpha$ and integrating from 0 to z we get

$$f(z) = \left\{ \frac{1}{\alpha} \int_0^z t^{-1-1/\alpha} p^{1/\alpha}(t) g^{1/\alpha}(t) h^{1/\alpha}(t) dt \right\}^{\alpha} \quad \alpha > 0.$$

Retracing the steps the converse follows. If $h(z) = z$ we deduce the integral representation of α-close-to convex function[2].

Similar representations can be got for other classes.

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