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A ROTATING MASS PARTICLE EMBEDDED IN A BIANCHI TYPE IX UNIVERSE

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An exact solution of Einstein's field equations corresponding to perfect fluid plus a pure radiation field is derived. The metric of the solution represents the exterior gravitational field of a Kerr particle emedded in a Bianchi type IX universe. In the absence of the source, the solution reduces to the solution describing a Bianchi type IX model. Many known solutions are derived as particular cases. The details of the solution are also discussed.

INTRODUCTION

Space-times admitting a 3 parameter group of automorphisms are important in discussing the cosmological models. The case where the group is simply transitive over the 3-dimensional, constant-time subspace is particularly useful for two reasons. First Bianchi has shown that there are only nine distinct sets of structure constants for groups of this type. Therefore we can use algebra to classify the homogeneous space-times. The second reason for the importance of Bianchi type space-times is the simplicity of the field equations. There is a large literature concerning various types of Bianchi space-times.

In the present paper we are going to deal with a Bianchi type IX space-time. Many relativists have taken keen interest in investigating Bianchi type IX space-times. One of the reasons behind this is the fact that the familiar solutions like Robertson Walker universe, the de-sitter universe, the Taub-NUT solution etc. are of Bianchi type IX.

The general axially symmetric Bianchi type IX metric can be expressed in the form

$$ds^2 = dt^2 - l^2 (d\psi + \cos\alpha\, d\beta)^2 - m^2 (d\alpha^2 + \sin^2 \alpha\, d\beta^2) \qquad ...(1.1)$$

where l and m are functions of time t. Assad and Soares[1] have considered the metric (1.1) in connection with Bianchi type IX cosmological models filled with perfect fluid.

Let us introduce the null co-ordinate u in place of the space-like co-ordinate ψ and a new time co-ordinate x defined by

$$u = \int \frac{a}{l}\, dt - a\psi, \quad x = \int \frac{l}{a}\, dt \qquad ...(1.2)$$

where a is a non-zero constant. It can be easily seen that the metric (1.1) can be expressed in terms u, α, β, x as

$$ds^2 = 2\,(du - a\cos\alpha\, d\beta)\, dx - m^2\,(d\alpha^2 + \sin^2\alpha\, d\beta^2) - l^2 a^{-2}\,(du - a\cos\alpha\, d\beta)^2. \qquad \ldots(1.3)$$

In (1.3), l and m are taken as functions of time co-ordinate x.

We propose to introduce inhomogeneity in the Bianchi type IX cosmological model described by the line-element (1.3). If the metric tensor in (1.3) is designated as g_{ik}, we shall introduce this inhomogeneity by adding a term $H\xi_i\,\xi_k$ to g_{ik} i.e., by taking the new metric tensor $\bar{g}_{ik}$ as representing what have been termed by Taub[3] as generalized Kerr-Schild metric

$$\bar{g}_{ik} = g_{ik} + H\xi_i\,\xi_k$$

where ξ_i is a null, geodetic and shear-free congruence in (1.3) and so

$$\xi_i\, dx^i = du - a\cos\alpha\, d\beta. \qquad \ldots(1.4)$$

The term $H\,\xi_i\,\xi_k$ added to g_{ik} of (1.3) essentially means adding a term $H\,(\alpha, u, x)$ to the coefficient of $(du - a\cos\alpha\; d\beta)^2$ in (1.3). Consequently we consider the line-element in the form

$$ds^2 = 2\,(du - a\cos\alpha\, d\beta)\, dx - m^2\,(d\alpha^2 + \sin^2\alpha\, d\beta^2) - \left[\frac{l^2}{a^2} + H\,(\alpha, u, x)\right](du - a\cos\alpha\, d\beta)^2 \qquad \ldots(1.5)$$

where l and m are functions of x alone. The metric (1.5) has now been written in the form what we have termed the Kerr-NUT metric[5] :

$$ds^2 = 2\,(du + g\sin\alpha\, d\beta)\, dx - 2L\,(du + g\sin\alpha\, d\beta)^2 - M^2\,(d\alpha^2 + \sin^2\alpha\, d\beta^2) \qquad \ldots(1.6)$$

with

$$g = g\,(\alpha),\ M = M\,(u, \alpha, x),\ L = L\,(u, \alpha, x).$$

The metric form (1.5) is effectively a special case of the Kerr-NUT metric (1.6). Therefore we have at our disposal the algebra and geometry of the metric (1.5) as developed there.

For the metric (1.5), let us introduce the tetrad

$$\theta^1 = du - a\cos\alpha\, d\beta,\ \theta^2 = m\, d\alpha$$

$$\theta^3 = m\sin\alpha\, d\beta,\ \theta^4 = dx - \frac{1}{2}\left[\frac{l^2}{a^2} + H\right]\theta^1. \qquad \ldots(1.7)$$

For the metric (1.6) and the tetrad

$$\theta^1 = du + g \sin\alpha \, d\beta, \; \theta^2 = M \, d\alpha, \; \theta^3 = M \sin\alpha \, d\beta$$

$$\theta^4 = dx - L\theta^1$$

the tetrad components $R_{(ab)}$ of the Ricci tensor have been calculated by Vaidya *et al.*[5]. Therefore $R_{(ab)}$ for the metric (1.5) with tetrad (1.7) can be obtained from the results given by Viadya *et al.*[5]. For the sake of brevity, we shall not list these $R_{(ab)}$ here. Therefore, nowonwards, we assume that the $R_{(ab)}$ for the metric (1.5) and the tetrad (1.7) are known.

2. The Field Equations

The material distribution filling the universe is assumed to be a mixture of perfect fluid and a pure radiation field. The Einstein's field equations for such a distribution are

$$R_{ik} - \tfrac{1}{2} g_{ik} R = -8\pi [(p + \rho) v_i v_k - p g_{ik}] - 8\pi\sigma w_i w_k - \Lambda g_{ik} \quad \ldots(2.1)$$

where $\sigma w_i w_k$ is the tensor arising out of flowing null radiation and Λ is the cosmological constant. The velocity vectors v^i and w^i satisfy

$$v_i v^i = 1, \; w_i w^i = 0, \; v_i w^i = 1. \quad \ldots(2.2)$$

The last condition in (2.2) is the normalizing condition for the null vector w_i.

One can express the field equations (2.1) in the following tetrad form

$$R_{(ab)} = \Lambda g_{(ab)} - 8\pi \sigma w_{(a)} w_{(b)} - 8\pi [(p + \rho) v_{(a)} v_{(b)} - \tfrac{1}{2} (\rho - p) g_{(ab)}]. \quad \ldots(2.3)$$

The metric (1.5) in the Cartan's frame (1.7) becomes

$$ds^2 = 2\theta^1 \theta^4 - (\theta^2)^2 - (\theta^3)^2 = g_{(ab)} \theta^a \theta^b \quad \ldots(2.4)$$

where $g_{(ab)}$ are the tetrad components of the metric tensor g_{ik}. For the metric (1.5) with the tetrad (1.7) we take the tetrad components $v_{(a)}$ and $w_{(a)}$ of v_i and w_i as

$$v_{(a)} = \left(\frac{1}{2\lambda}, 0, 0, \lambda \right), \; w_{(a)} = \left(\frac{1}{\lambda}, 0, 0, 0 \right) \quad \ldots(2.5)$$

λ being a function of co-ordinates. It is painless to verify that v_i and w_i given by (2.5) satisfy the conditions (2.2). In view of (2.4) and (2.5) the field equations (2.1) imply

$$R_{(12)} = 0, \; R_{(13)} = 0 \quad \ldots(2.6)$$

$$8\pi p = \Lambda - R_{(14)} \quad \ldots(2.7)$$

$$8\pi \rho = -\Lambda - R_{(14)} - 2 R_{(22)} \quad \ldots(2.8)$$

$$\lambda^2 = R_{(44)}/2\,[R_{(44)} + R_{(22)}] \qquad \ldots(2.9)$$

$$16\pi\,\sigma = R_{(14)} + R_{(22)} - \frac{R_{(11)}\,R_{(44)}}{R_{(14)} + R_{(22)}} \qquad \ldots(2.10)$$

where $R_{(ab)}$ are given by expressions given in paper by Vaidya *et al.*[5] corresponding to the metric (1.5).

3. A Mass Particle in Bianchi Type IX Universe

Equations (2.6) are

$$H_{xy} + \frac{a}{m^2} H_u = 0, \; H_{xu} - \frac{a}{m^2} H_y = 0. \qquad \ldots(3.1)$$

Here and in what follows a suffix indicates partial derivative $\left(\text{e.g. } l_x = \frac{\partial l}{\partial x}, \; H_{xy} = \frac{\partial^2 H}{\partial x.\partial y} \text{ etc.}\right)$ The variable y is defined by $y = a \log \operatorname{cosec} \alpha$. It is not hard to verify that the solution of the eqns. (3.1) is

$$H = 2\,F(r) \operatorname{cosec}^2 \alpha, \; r = u - \int \frac{a^2}{2m^2}\, dx$$

$$\frac{d^2F}{dr^2} + \frac{4F}{a^2} = 0. \qquad \ldots(3.2)$$

The four equations (2.7), (2.8), (2.9) and (2.10) are sufficient to determine the pressure p, the density ρ, λ^2 and the radiation density σ. These four parameters are given by

$$8\pi p = -\frac{l^2}{a^2}\left\{\frac{l_{xx}}{l} + \frac{l_x^2}{l^2} + 2\,\frac{l_x}{l}\,\frac{m_x}{m} + A + \frac{a^2}{2m^4}\right\} - 2AF \operatorname{cosec}^2\alpha + \Lambda \qquad \ldots(3.3)$$

$$8\pi\,(\rho + p) = -\,2 \operatorname{cosec}^2 \alpha \left\{\frac{F_r\, a^2 m_x}{m^3} + \frac{a^2 F}{2m^4} - \frac{2Fm_x^2}{m^2}\right\}$$

$$-\frac{2l^2}{a^2}\left\{\frac{l_{xx}}{l} + \frac{l_x^2}{l^2} + \frac{3a^2}{4m^4} - \frac{a^2}{l^2 m^2} - \frac{m_x^2}{m^2}\right\} \qquad \ldots(3.4)$$

$$\lambda^2 = \frac{-A}{4\pi\,(p + \rho)} \qquad \ldots(3.5)$$

$$8\pi\sigma = -\,2\pi\,(p + \rho) + \frac{A}{4\pi\,(p + \rho)}\left\{2A\left(\frac{l^2}{2a^2} + F \operatorname{cosec}^2 \alpha\right)^2\right.$$

$$\left. + \frac{2F}{m^2} \operatorname{cosec}^2 \alpha + 2F_r \operatorname{cosec}^2 \alpha\, \frac{m_x}{m}\right\} \qquad \ldots(3.6)$$

where A is defined by

$$A = \frac{m_{xx}}{m} - \frac{a^2}{4m^4}. \qquad ...(3.7)$$

The co-ordinate componets v_i of the flow vector of the fluid can be obtained by

$$v_i = e_i^{(a)} v_{(a)}, \quad e_i^{(a)} dx^i = \theta^a.$$

We have found that

$$v_i = \left(- \frac{B}{2\lambda}, 0, \frac{aB}{2\lambda} \cos \alpha, \lambda \right) \qquad ...(3.8)$$

where λ is given by (3.5) and B is defined by

$$B = \frac{l^2 \lambda^2}{a^2} - 1 + 2F \lambda^2 \operatorname{cosec}^2 \alpha. \qquad ...(3.9)$$

Here we have named the co-ordinates as $u = x^1$, $\alpha = x^2$, $\beta = x^3$, $x = x^4$. We have also verified that the flow vector v^i, given by (3.8) is not irrotational. By a lengthy but straightforward calculation, we can find the angular velocity vector Ω^i of the flow vector v_i. The components of Ω^i are given by

$$\Omega^1 = \frac{aB}{2m^2} + \frac{2a}{m^2} F \lambda^2 \cot^2 \alpha \operatorname{cosec}^2 \alpha$$

$$\Omega^2 = - \frac{a}{m^2} \cot \alpha \operatorname{cosec}^2 \alpha \, \lambda^2 F_r, \quad \Omega^3 = \frac{2\lambda^2}{m^2} \cot \alpha \operatorname{cosec}^3 \alpha . F$$

$$\Omega^4 = - \frac{a B^2}{4\lambda^2 m^2} \qquad ...(3.10)$$

where B is given by (3.9).

The final form of the metric of our solution is

$$ds^2 = 2 (du - a \cos \alpha \, d\beta) \, dx - m^2 (d\alpha^2 + \sin^2 \alpha \, d\beta^2)$$
$$- \left[\frac{l^2}{a^2} + 2F(r) \operatorname{cosec}^2 \alpha \right] (du - a \cos \alpha \, d\beta)^2 \qquad ...(3.11)$$

with

$$\frac{d^2 F}{dr^2} + \frac{4F}{a^2} = 0.$$

We can take the solution of this equation as $F = m_0 \sin (2r/a)$ where m_0 is an arbitrary constant. When $l = m$, the metric (3.11) describes the field of a Kerr particle embedded in the closed Robertson-Walker univese discussed by Vaidya[4]. Moreover if $l = m$ = constant $= a$, we recover the metric describing the field of a Kerr particle embedded in static Einstein Universe [see Patel and Vaidya[2]]. We know that $\Lambda \neq 0$

for Einstein universe. If we take $\Lambda = 0$ in the field equations (2.1), then we cannot get the above mentioned solution as a particular case. This is the main reason for introducing the cosomological constant Λ in (2.1). From the above analysis we can say that the metric (3.11) describes the exterior gravitational field of a rotating mass particle in the background of a Bianchi type IX universe.

In the absence of source (i.e when $F = 0$), the vanishing of the radiation density σ implies

$$\frac{l_{xx}}{l} + \frac{l_x^2}{l^2} + \frac{a^2}{m^4} - \frac{a^2}{l^2m^2} - \frac{m_{xx}}{m} - \frac{m_x^2}{m^2} = 0. \quad \text{.. (3.12)}$$

In this case the pressure p and the density ρ are given by

$$8\pi\, p = \Lambda - \frac{l^2}{a^2}\left[2A + \frac{a^2}{l^2m^2} - \frac{a^2}{4m^4} + \frac{m_x^2}{m^2} + \frac{2l_x\, m_x}{lm}\right] \quad \text{...(3.13)}$$

$$8\pi\,(p + \rho) = -\frac{2l^2}{a^2}A. \quad \text{...(3.14)}$$

The equations (3.12), (3.13) and (3.14) are the governing equations for a Biahchi type IX cosmological model filled with perfect fluid. In this case $\lambda^2 = a^2/l^2$ and the rotation vector Ω^i becomes zero. Therefore the perfect fluid filling the Bianchi type IX universe is irrotational. It should be noted that the functions l and m in (3.11) are now not arbitrary. But they satisfy the differential equation (3.12). Thus the introduction of inhomogeneity in (1.3) gives rise to the rotation of the flow vector and the radiation density σ.

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