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SUBMANIFOLDS OF A FINSLER MANIFOLD I*

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Gauss-Codazzi equations in a submanifold of a Finsler manifold have been studied by various authors. In this paper we study a submanifold of a Finsler manifold admitting a C^p Γ-connection defined by Hōjō[4] and obtain many kinds of Gauss-Codazzi equations using various derivatives and a method similar to Rastogi[6]. We also obtain the relationships between the Riemannian curvatures and the Ricci tensors of the submanifold of a Finsler manifold of Finsler manifold and the enveloping manifold.

1. INTRODUCTION

Consider a Finsler manifold $(F^m, \bar{L})$ of coordinates u^A $(A = 1, ..., m)$ and metric tensor g_{AB} (u, v) imbedded in a Finsler manifold (F^n, L) of coordinates x^i $(i = 1, ...,n)$ and metric tensor g_{ij} (x, y). Let $B^i_A = \partial x^i/\partial u^A$ and $\bar{L}(u^A, v^B) = L(x^i(u^A), B^i_A(u^B)\, v^A)$, then we have[7] [p. 156, eqn. (2.6)].

$$g_{AB}(u, v) = g_{ij}(x, y)\, B^{ij}_{AB}. \qquad ...(1.1)$$

Let $B^C_i = g^{AC} g_{ij} B^j_A$ be the inverse of B^i_A and let $N^i_{(a)}$, $(a = m + 1,..., n)$ be $n - m$ linearly independent unit vectors normal to F^m, which can be chosen arbitrarily in an infinite number of ways such that

$$B^i_A B^C_i = \delta^C_A, \quad B^i_A B^A_j = \delta^i_j - \sum_a^n N^i_{(a)} N_{(a)j} \qquad ...(1.2)$$

$$B^i_A N^i_{(a)i} = 0, \; N^i_{(a)} g^{ij} N_{(a)j} \qquad ...(1.3)$$

and

$$g_{ij} N_{(a)} N^j_{(b)} = \delta_{ab}. \qquad ...(1.4)$$

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Corresponding to $\phi^{(p)}(x, y)$, p being a real number, $(p \neq 1)$, defined by Hōjō[4]:

$$\phi^{(p)} = p^{-1} L^p \ (p \neq 0); \ \phi^{(0)} = \log L \qquad \ldots(1.5)$$

we also have

$$\phi_i^{(p)} \ L^{(p-1)} \ l_i, \ l_i = L_{.i} = \partial L/\partial y^i \qquad \ldots(1.6)$$

$$\phi_{ij}^{(p)} = L^{(p-2)} (g_{ij} + (p - 2) \ l_i \ l_j) \qquad \ldots(1.7)$$

$$\phi_{ijk}^{(p)} = L^{(p-2)} [2 \ C_{ijk} + (p - 2) \ L^{-1} (h_{jk} \ l_j + h_{ij} \ l_k + h_{ki} \ l_j + (p - 1) \ l_j \ l_j \ l_k)] \qquad \ldots(1.8)$$

$$C_{ij}^{(p)r} = C_{ij}^r + ((p - 2)/2L) \ (\delta_i^r \ l_j + \delta_j^r \ l_i + h_{ij} \ l^r \ (p - 1)^{-1} - l_i \ l_j \ l^r). \qquad \ldots(1.9)$$

$$R_{ijk}^{(p)r} = R_{ijk}^r + ((p - 2)/2L) \ (R_{ijk}^r \ l_i + R_{ijk} \ l^r \ (p - 1)^{-1}) \qquad \ldots(1.10)$$

$$P_{ijk}^{(p)r} = P_{ijk}^r + ((p - 2)/2L) \ (P_{jk}^r \ l_i + P_{ijk} \ l^r \ (p - 1)^{-1}) \qquad \ldots(1.11)$$

and

$$S_{ijk}^{(p)r} = S_{ijk}^r - ((p - 2)^2/4(p - 1) \ L^2) \ (h_{ij} \ h_k^r - j \mid k) \qquad \ldots(1.12)$$

where $-j \mid k$ means change of indices j and k and substraction and $h_{ij} = g_{ij} - l_i \ l_j$.

If we use $|$, $|$, $\|$, $\|$ respectively for h −, v-covariant derivatives based on $C\Gamma$ and $C^p\Gamma$-connections then for an arbitrary tensor u_i^j we have[4]

$$u_{i\|k}^j = u_{i,k}^j, \ u_i^j \|_k = u_i^j \mid_k + \sigma_{mk}^{(p)j} \ u_i^m - \sigma_{ik}^{(p)m} \ u_m^j \qquad \ldots(1.13)$$

where

$$\sigma_{ij}^{(p)m} = ((p - 2)/2L) \ (\delta_i^m \ l_j + \delta_j^m \ l_i + h_{ij} \ l^m \ (p - 1)^{-1} - l_i \ l_j \ l^m). \qquad \ldots(1.14)$$

2. Connection Parameters

For a submanifold F^m corresponding to (1.5) we can write

$$\bar{\phi}^{(p)} = p^{-1} (\bar{L})^p, \ p \neq 0, \ \bar{\phi}^{(0)} = \log \bar{L}. \qquad \ldots(2.1)$$

Analogous to (1.6), (1.7) and (1.8) from (2.1) we can obtain

$$\bar{\phi}^{(p)}_{A} = (\bar{L})^{(p-1)}\, l_A = \phi^{(p)}_{i}\; B^{i}_{A}\,, l_A = \partial \bar{L}/\partial v^A = l_i\; B^{i}_{A} \qquad \ldots(2.2)$$

$$\bar{\phi}^{(p)}_{AB} = (\bar{L})^{(p-2)}\,(g_{AB} + (p-2)\, l_A\, l_B) = \phi^{(p)}_{ij}\; B^{ij}_{A_B} \qquad \ldots(2.3)$$

and

$$\bar{\phi}^{(p)}_{ABC} = (\bar{L})^{(p-2)}\,[2\, C_{ABC} + (p-2)\,(\bar{L})^{-1}\,(h_{AB}\, l_C + h_{BC}\, l_A + h_{CA}\, l_B + (p-2)\, l_A\, l_B\, l_C)] = \phi^{(p)}_{ijk}\;\; B^{ijk}_{ABD} \qquad \ldots(2.4)$$

where

$$h_{AB} = g_{AB} - l_A\, l_B, C_{ABD} = C_{ijk}\, B^{ijk}_{ABD} \qquad \ldots(2.4)$$

Multiplying (1·9) by $B^{ij}_{KB}\; B^{D}_{r}$ and using $C^{r}_{ij}\; B^{ij}_{AB}\; B^{D}_{r} = C^{D}_{A_B}$ and $C^{(p)D}_{AB}$

$= C^{(p)r}_{ij}\;\; B^{ij}_{AB}\; B^{D}_{r}$ we obtain

$$C^{(p)D}_{AB} = C^{D}_{AB} + ((p-2)/2L)\;\; (\delta^{D}_{A}\; l_B + \delta^{D}_{B}\, l_A - l_B\, l^D + h_{AB}\, l^D\, (p-1)^{-1}). \qquad \ldots(2.5)$$

Let $F^{A}_{BC}\,(u, v)$ and $\hat{F}{}^{A}_{BC}\,(u, v)$ respectively denote the induced and intrinsic connection parameters based on $C\Gamma$-connection then they are given by[7] [p. 214, eqn. (9.25)]

$$\hat{F}{}^{A}_{BC} = F^{A}_{BC} + \Lambda^{A}_{BC}\,, \qquad \ldots(2.6)$$

where

$$F^{A}_{BC} = B^{A}_{i}\;\; (B^{i}_{BC} + F^{i}_{jk}\; B^{jk}_{BC}) \qquad \ldots(2.7)$$

and

$$\Lambda_{ABC} = \sum_{a} [(M_{(a)BC}\, \tilde{\Omega}_{(a)AD} + M_{(a)AC}\;\; \tilde{\Omega}_{(a)BD} - M_{(a)AB}\;\; \tilde{\Omega}_{(a)CD})\; v^D - (M_{(a)EA}\, C^{E}_{BC} + M_{(a)EB}\;\; C^{E}_{AC} - M_{(a)EC}\, C^{E}_{AB})\;\; \tilde{\Omega}_{(a)AF}\; v^D\, v^F] \qquad \ldots(2.8)$$

such that

$$C_{ijk}\; N^{k}_{(a)} = M_{(a)ij}, \quad M_{(a)ij}\;\; N^{i}_{(b)} = M_{(ab)j}, \quad M_{(a)ij}\;\; B^{ij}_{AB}$$

(*equation continued on p.* 899)

$$= M_{(a)AB}, \quad M_{(a)|j}\, y^j = 0, \; M_{(ab)j}\; B^j_A = M_{(ab)A}. \qquad \ldots(2.9)$$

As we can write $dx^i = B^i_A\, du^A$, $y^i = B^i_A\, v^A$, therefore we can get

$$y^i\, N_{(a)i} = 0, \; y^i\, N_{(a)i\cdot j} = -N_{(a)j}, \; N_{(a)i\cdot j}\, y^i y^j = 0. \qquad \ldots(2.10)$$

From (1.9) we can obtain

$$C^{(p)i}_{jk}\, y^j = ((p-2)/2)\; \delta^i_k\,, \quad C^{(p)i}_{jk}\, y^j\, y^k = ((p-2)/2)\, y^i,$$

$$2\; C^{(p)}_{ijk}\; y^j\, y^k\, y^i = (p-2)\, L^2 \qquad \ldots(2.11)$$

where

$$C^{(p)}_{ijk} := g_{ih}\; C^{(p)h}_{jk}.$$

Further from (1.9) we can also obtain

$$M^{(p)}_{(a)jk} = M_{(a)jk} + ((p-2)/2L)\,(N_{(a)j}\; l_k + N_{(a)k}\; l_j) \qquad \ldots(2.12)$$

where

$$C^{(p)i}_{jk}\; N_{(a)i} = M^{(p)}_{(a)jk}\,.$$

Equation (2.12) easily gives

$$M^{(p)}_{(a)ik}\; B^{jk}_{AB} = M_{(b)AB}, \; M^{(p)}_{(a)jk}\; y^k = ((p-2)/2)\; N_{(a)j}. \qquad \ldots(2.13)$$

3. Mixed Covariant Differentiations

If T^i_A is an arbitrary tensor its mixed covariant differentation corresponding to induced and intrinsic connection parameters is given by Rund[8,9] as follows :

$$T^i_{A\cdot B} = \partial_B\, T^i_A \; - T^i_{A\cdot C}\; F^C_{BD}\, v^D + T^r_A \;\; B^j_B \;\; F^i_{rj} - T^i_D \;\; F^D_{AB} \qquad \ldots(3.1)$$

$$T^i_{A|B} = T^i_{A/B} - T^i_{A\cdot C}\; \Lambda^C_{BD}\, v^D - T^i_D\; \Lambda^D_{AB}. \qquad \ldots(3.2)$$

Corresponding to (3.1) and (3.2) we have

$$I^i_{AB} = B^i_{AB} - B^i_D \;\; F^D_{AB} + F^i_{hk}\; B^{hk}_{AB} = \sum_a N^i_{(a)}\; \tilde{\Omega}_{(b)AB} \qquad \ldots(3.3)$$

$$J^i_{AB} = B^i_{A|B} = I^i_{AB} - B^i_D \Lambda^D_{AB} \quad \ldots(3.4)$$

and

$$N^i_{(a)B} = - \tilde{\Omega}_{(a)AB} g^{AC} B^i_C + E^i_{(a)j} I^j_{DB} v^D \quad \ldots(3.5)$$

where

$$E^i_{(a)j} = \sum_b N^i_{(b)} M_{(ab)j} - 2M^i_{(a)j} .$$

We now define here the mixed covariant differentiation based on C^p Γ-connection as follows :

$$T^i_{A;B} := T^i_{A\cdot B} + T^r_A C^{(p)i}_{rj} B^j_B - T^i_D C^{(p)D}_{AB} . \quad \ldots(3.6)$$

From (3.6) it is easy to obtain

$$l^i_{;A} = (p/2L) B^i_A - l^i l_A/L, \; y^i_{;A} = (p/2) B^i_A \quad \ldots(3.7)$$

which gives

$$l^i_{;A} N_{(a)i} = 0, \; y^i_{;A} N_{(a)i} = 0, \; l^i_{;A} B^A_i = (pm - 2)/2L,$$

$$y^i_{;A} B^A_i = (pm/2), \; l^i N_{(a)i;A} = 0, \; y^i N_{(a)i;A} = 0. \quad \ldots(3.8)$$

Also from

$$g_{ij}\|_k = - (\sigma^{(p)}_{ijk} + \sigma^{(p)}_{jik}), \; g^{ij}\|_k = (g^{mj} \sigma^{(p)i}_{mk} + g^{mi} \sigma^{(p)j}_{mk}) \quad \ldots(3.9)$$

where

$$\sigma^{(p)}_{ijk} = g_{jr} \sigma^{(p)r}_{ik},$$

$$g_{ij;C} = - B^k_C (\sigma^{(p)}_{ijk} + \sigma^{(p)}_{jik}), \; g^{ij}{}_{;C} = B^k_C (g^{mj} \sigma^{(p)i}_{mk} + g^{mi} \sigma^{(p)j}_{mk}). \quad \ldots(3.10)$$

If we write $B^i_{A;B} = U^i_{AB}$ from (3.6) we can obtain

$$U^i_{AB} = - B^i_D C^{(p)D}_{AB} + B^{jk}_{AB} C^{(p)i}_{jk} \quad \ldots(3.11)$$

which satisfies

$$U^i_{AB} B^D_i = 0, \; U^i_{AB} N_{(a)i} = M_{(a)AB}, \; U^i_{AB} l_i = 0. \quad \ldots(3.12)$$

From the first equation of (3.12) it can be observed that U^i_{AB} can be treated as a vector of F^n normal to F^m, therefore it can be expressed as

$$U^i_{AB} = \sum_a M_{(a)AB}\ N^i_{(a)} \ . \qquad \ldots(3.13)$$

If we differentiate B^D_i with respect to (3.6) we get

$$B^D_{i;A} = B^E_i\ C^{(p)D}_{EA} - B^D_j\ C^{(p)j}_{ik}\ B^k_A \qquad \ldots(3.14)$$

which satisfies by virtue of (3.13)

$$B^D_{i;A}\ B^i_F = 0, \quad B^D_{i;A}\ N^i_{(a)} = -\ g^{DF}\ M_{(a)AF},\ l^i\ B^D_{i;A} = 0. \qquad \ldots(3.15)$$

To calculate $N^i_{(a);B}$ we assume that it is expressible as

$$N^i_{(a);B} = B^i_A\ T^A_{(a)B} + \sum_b C_{(ab)B}\ N^i_{(b)} \qquad \ldots(3.16)$$

where $T^A_{(a)B}$ and $C_{(ab)B}$ are parameters to be determined.

Differentiating first equation of (1.3) by virtue of (3.6) we obtain the following equation

$$g_{ij}\ B^i_A\ N^j_{(a);B} = -\ M_{(a)AB} + (\sigma^{(p)}_{ijk} + \sigma^{(p)}_{jik})\ B^{kl}_{BA}\ N^j_{(a)} \ . \qquad \ldots(3.17)$$

Similarly by differentiating (1.4) we get

$$g_{ij}\ N^i_{(a);B}\ N^j_{(b)} = \sigma^{(p)l}_{ij}\ N^i_{(a)}\ B^j_B\ N_{(b)l} \qquad \ldots(3.18)$$

whereas from (3.16) we can obtain

$$g_{ij}\ B^i_A\ N^j_{(a);B} = T^C_{(a)B}\ g_{AC},\ g_{ij}\ N^i_{(a);B}\ N^j_{(b)} = C_{(ab)B}. \qquad \ldots(3.19)$$

It is easy to see that

$$(\sigma^{(p)}_{ijk} + \sigma^{(p)}_{jik})\ B^{kl}_{BA}\ N^j_{(a)} = 0,\ \sigma^{(p)i}_{ij}\ B^j_B\ N^i_{(a)}\ N_{(b)l} = ((p-2)/2L)\ l_B\ \delta_{ab} \qquad \ldots(3.20)$$

therefore from (3.17), (3.18) and (3.19) we get

$$T^C_{(a)B}\ g_{AC} = -\ M_{(a)AB},\ C_{(ab)B} = ((p-2)/2L)\ l_B\ \delta_{ab}. \qquad \ldots(3.21)$$

Substituting from (3.21) in (3.16) we get

$$N^i_{(a);B} = - M^B_{(a)B} \quad B^i_A + ((p-2)/2L)\, I_B\, N^i_{(a)} \,. \qquad \ldots(3.22)$$

If $T^i_{A\,//B}$ denotes the mixed covariant differentiation corresponding to Berwald's connections, then we can write

$$V^i_{AB} = B^i_{A|B} = \sum_a N^i_{(a)}\, \tilde{\Omega}_{(a)AB} + B^{hk}_{AB}\, C^i_{hk|r}\, y^r$$

$$- B^i_D\, (\Lambda^D_{AB} + C^D_{AB,E}\, v^E) \qquad \ldots(3.23)$$

and

$$N^i_{(a)//B} = N^i_{(a),B} + C^i_{rh|j}\, y^j\, N^r_{(a)}\, B^h_B \,. \qquad \ldots(3.24)$$

4. Generalised Gauss-Codazzi Equations

Let us assume that we have a tensor ω^i_{AB} satisfying

$$\omega^i_{AB}\, N_{(a)i} = 0. \qquad \ldots(4.1)$$

For some tensor Q^D_{AB} we can now express ω^i_{AB} as follows :

$$\omega^i_{AB} = B^i_D\, Q^D_{AB} \,. \qquad \ldots(4.2)$$

From equation (4.2) we can obtain

$$\omega^i_{AB;C} = U^i_{DC}\, Q^D_{AB} + B^i_D\, Q^D_{AB;C}, \qquad \ldots(4.3)$$

which by virtue of (3.12) gives

$$\omega^i_{AB;C}\, B^E_i = Q^E_{AB;C}, \quad \omega^i_{AB;C}\, N_{(a)i} = M_{(a)DC}\, Q^D_{AB} \,. \qquad \ldots(4.4)$$

On the basis of this tensor ω^i_{AB} we now define the following tensor

$$L^i_{AB} := - B^i_D\, C^{(p)D}_{AB} + B^{jk}_{AB}\, C^{(p)i}_{jk} + \omega^i_{AB}. \qquad \ldots(4.5)$$

From eqn. (4.5) with the help of (3.6) we get

$$(L^i_{AB;C} - L^i_{AC;B})\, B^E_i = (Q^E_{AB;C} - Q^E_{AC;B}) + 2S^{(p)i}_{jkh}\, B^{jkh}_{ABC}\, B^E_i - 2\, S^{(p)E}_{ABC}$$

(equation continued on p. 903)

$$+ \sum_a (M_{(a)AC}\ M^E_{(a)B}\ M_{(a)AB}\ M^E_{(a)C})$$

and

$$(L^l_{AB;C} - L^l_{AC;B})\ N_{(a)l} = M_{(a)DC}\ (Q^D_{AB} - C^{(p)D}_{AC})\ M_{(a)DB}\ (Q^D_{AC} - C^{(p)D}_{AC}$$

$$+ \sum (M_{(b)AC}\ M^{(p)}_{(ab)B} - M_{(b)AB}\ M^{(p)}_{(ab)C})$$

$$+ 2\ S^{(p)l}_{jkh}\ B^{jkh}_{ABC}\ N_{(a)l}$$

where $S^{(p)E}_{ABD}$ is a tensor of F^m similar to (1.12) and $M^{(p)}_{(ab)C} = M_{(ab)C}$ $+ ((p-2)/2L)\ l_C\ \delta_{ab}$.

Let $V^l_{(a)}$ be the contravariant components of a unit vector tangential to the curves of the congruence $V_{(a)}$, then we have[5]

$$V^l_{(a)} = t^A_{(a)}\ B^l_A + \sum_b d_{(ab)}\ N^l_{(b)} \qquad ...(4.6)$$

where $t^A_{(a)}$ and $d_{(ab)}$ are parameters.

Let L^l_{AB} be treated as a vector of the enveloping manifold in the direction of the congruence, then we can write

$$L^l_{AB} = \sum_a \Omega_{(a)AB}\ V^l_{(a)} \qquad ...(4.7)$$

for some symmetric tensor $\Omega_{(a)AB}$ of F^m.

From eqn. (4.6) by virtue of (3.11) and (3.22) we can obtain on simpiification

$$V^l_{(a);B} = X^D_{(a)B}\ B^l_D + \sum_b Y_{(ab)B}\ N^l_{(b)} \qquad ..(4.8)$$

where

$$X^D_{(a)B} = t^D_{(a),B} - \sum_b d_{(ab)}\ M^D_{(b)B} \qquad ...(4.9)$$

and

$$Y_{(ab)B} = t^A_{(a)} M_{(b)AB} + d_{(ab);B} + ((p-2)/2L)\ I_B\ d_{(ab)}. \qquad ...(4.10)$$

From eqn. (4.7) by virtue of (4.8), (4.9) and (4.10) one can easily obtain

$$L^{l}_{AB;C} - L^{l}_{AC;B} = \Big[B^{l}_{D} \sum_{a} (\Omega_{(a)AB;C}\, t^{D}_{(a)} + \Omega_{(a)AB}\, X^{D}_{(a)C}) + \sum_{a} \sum_{b} N^{l}_{(b)} \, (\Omega_{(a)AB;C}\, d_{(ab)} + \Omega_{(a)AB}\, Y_{(ab)C}) \Big] - B \mid C.$$

Multiplying this equation by B^{E}_{i} and $N_{(c)l}$ respectively and comparing with similar equations obtained earlier, we get by virtue of (1.2), (1.3) and (1.4) on simplification

$$S^{(p)}_{AFBC} = S^{(p)}_{rhjk}\; B^{rhjk}_{AFBC} + \tfrac{1}{2} [g_{DF}\, Q^{D}_{AB;C} + \sum_{a} (M_{(a)AC}\, M_{(a)BF} - t_{(a)F}\, \Omega_{(a)AB;C} - \Omega_{(a)AB}\, X_{(a)CF})] - B \mid C \qquad \ldots(4.11)$$

and

$$S^{(p)}_{rhjk}\; B^{rjk}_{ABC}\; N^{h}_{(c)} + \tfrac{1}{2} [M_{(c)DC}\, (Q^{D}_{AB} - C^{(p)D}_{AB}) - \sum_{a} (d_{(ac)}\, \Omega_{(a)AB;C} + \Omega_{(a)AB}\, Y_{(ac)C} + M_{(a)AB}\, M^{(p)}_{(ac)C})] - B \mid C = 0. \qquad \ldots(4.12)$$

Equations (4.11) and (4.12) are the generalized Gauss-Codazzi equations of the first kind based on the connectian of Hōjō[4].

5. Gauss-Codazzi Equations of other Kinds

If in place of (3.6) we use (3.1) we can obtain from (4.6)

$$V^{l}_{(a),B} = \amalg^{D}_{(a)B}\; B^{l}_{D} + \sum_{b} Z_{(ab)B}\, N^{l}_{(b)} \qquad \ldots(5.1)$$

where

$$\amalg^{D}_{(a)B} = t^{D}_{(a),B} - \sum_{b} d_{(ab)} \, (\widetilde{\Omega}_{(b)CB}\, g^{CD} + E_{lhk}\, g^{AD}\, B^{lk}_{AB}\, N^{h}_{(b)}) \qquad \ldots(5.2)$$

and

$$Z_{(ab)B} = t^{D}_{(a)}\, \widetilde{\Omega}_{(b)DB} + d_{(ab),B} + \sum_{e} d_{(ae)}\, N_{(be)B} \qquad \ldots(5.3)$$

where

$$E_{lhk} = g_{lh,k}, \quad N_{(bd)B} = N^{l}_{(b),B}\, N_{(d)l}.$$

Differentiating (4.5) and (4.7) with respect to covariant differentiation given by (3.1) we obtain by virtue of (5.1) on simplification

$$\Big[B^{l}_{D}\ C^{(p)D}_{AC'B} + C^{(p)i}_{jk'h}\ B^{jkh}_{ABC} + \omega^{l}_{AB'C} + \sum_{a} \Big\{ C^{(p)D}_{AC}\ \widetilde{\Omega}_{(a)DB}\ N^{l}_{(a)} + C^{(p)l}_{jk}\ N^{j}_{(a)}\ B^{k}_{B}\ \widetilde{\Omega}_{(a)AC} - B^{l}_{D}\ (\Omega_{(a)AB} \prod^{D}_{(a)C} + t^{D}_{(a)}\ \Omega_{(a)AB,C}) - \sum_{b} N^{l}_{(b)}\ (\Omega_{(a)AB,C}\ d_{(ab)} + Z_{(ab)C}\ \Omega_{(a)AB}) \Big\} \Big] - B \mid C = 0. \qquad \ldots(5.4)$$

Now using

$$(P^{(p)l}_{hki} + C^{(p)l}_{kt}\ P^{t}_{hi} - C^{(p)l}_{jk|h}) - h \mid k = 0 \qquad \ldots(5.5)$$

$$g_{EF}\ B^{E}_{i}\ \omega^{i}_{AB'C} = g_{DF}\ Q^{D}_{AB'C};\quad \omega^{l}_{AB'C}\ N_{(c)l} = \widetilde{\Omega}_{(c)DC}\ Q^{D}_{AB} \qquad \ldots(5.6)$$

and a relation similar to (5.5) for F^m we obtain in analogy to (4.11) and (4.12) the following relations

$$\Big[P^{(p)E}_{BCA} + C^{(p)E}_{CD}\ P^{D}_{BA} + Q^{E}_{AB'C} + B^{jkh}_{ABC}\ B^{E}_{i}\ (P^{(p)l}_{hkl} + C^{(p)j}_{kt}\ P^{t}_{hl}) + \sum_{a} (M^{(p)E}_{(a)B}\ \widetilde{\Omega}_{(a)AC} - t^{E}_{(a)}\ \Omega_{(a)AB,C} - \Omega_{(a)AB} \prod^{E}_{(a)C}) \Big] - B \mid C = 0 \qquad \ldots(5.7)$$

and

$$\Big[B^{jkh}_{ABK}\ N_{(b)l}\ (P^{(p)l}_{hkl} + C^{(p)i}_{kt}\ P^{t}_{hl}) + (C^{(p)D}_{AC} - Q^{D}_{AC})\ \widetilde{\Omega}_{(b)DB} + \sum_{a} (M^{(p)}_{(ab)B}\ \widetilde{\Omega}_{(a)AC} - d_{(ab)}\ \Omega_{(a)AB,C} - Z_{(ab)C}\ \Omega_{(a)AB}) \Big] - B \mid C = 0. \qquad \ldots(5.8)$$

Equations (5.7) and (5.8) are the generalized Gauss-Codazzi equations of second kind based on the curvatare tensors $P^{(p)l}_{hkl}$ of F^n and $P^{(p)E}_{ABC}$ of F^m.

If in place of covariant derivatives given by (3.1) we use the covariant derivative given by (3.2) we obtain in analogy to (5.7) and (5.8), the generalized Gauss-Codazzi equations of third kind in the following from

$$\Big[{}^{*}P^{(p)D}_{BCA} + \Lambda^{E}_{DC}\ (C^{(p)D}_{AB} - Q^{D}_{AB}) - C^{(p)E}_{DB}\ \Lambda^{D}_{AC} + Q^{E}_{AB|C}$$

(*equation continued on p.* 906)

$$+ B^{jkh}_{ABC}\ B^{E}_{i}\ (P^{(p)l}_{hkj} + C^{(p)i}_{kt} P^{t}_{hj}) + \sum_{a} (M^{(p)E}_{(a)B}\ \widetilde{\Omega}_{(a)AC}$$

$$- t^{E}_{(a)} \Omega_{(a)AB|C} - \Omega_{(a)AB} \coprod^{E}_{(a)C}) \Big] - B \mid C = 0 \qquad \ldots(5.9)$$

and

$$\Big[B^{jkh}_{ABC} N_{(b)l}\ (P^{(p)l}_{hkj} + C^{(p)i}_{kt} P^{t}_{hj}) + \widetilde{\Omega}_{(b)DB}\ (C^{(p)D}_{AC} - Q^{D}_{AC})$$

$$- M^{(p)}_{(b)D_B}\ \Lambda^{D}_{AC} + \sum_{a} (M^{(p)}_{(ab)_B}\ \widetilde{\Omega}_{(a)AC} - d_{(ab)}\ \Omega_{(a)AB|C}$$

$$- Z_{(ab)C}\ \Omega_{(a)AB})] - B \mid C = 0. \qquad \ldots(5.10)$$

where

$$^{*}P^{(p)D}_{BCA} := P^{(p)D}_{BCA} + C^{(p)D}_{CE}\ P^{E}_{AB} - C^{(p)D}_{AC,E}\ \Lambda^{E}_{BF}\ v^{F} - C^{(p)D}_{EC}\ \Lambda^{E}_{AB} \qquad \ldots(5.12)$$

Similar to (5.9) and (5.10) on the basis of (3.23) and (3.24) we can analogously obtain

$$\Big[{}^{*}C^{(p)D}_{AB}\ (\Lambda^{E}_{DC} + C^{E}_{DC,F}\ v^{F}) - B^{hk}_{DC}\ B^{E}_{i}\ C^{i}_{hk|j}\ y^{j}\ {}^{*}C^{(p)D}_{AB}$$

$$- {}^{*}C^{(p)E}_{AB|C} + C^{(p)l}_{jk}\ B^{E}_{i}\ B^{k}_{B}\ V^{j}_{AC} + B^{jkh}_{ABC}\ B^{E}_{i}\ C^{(p)l}_{jk||h}$$

$$- \sum_{a} t^{E}_{(a)} \Omega_{(a)AB||C} + \Omega_{(a)AB} (\coprod^{E}_{(a)C} + C^{l}_{rh|j}\ y^{j}\ B^{E}_{i} \qquad \ldots(5.12)$$

$$B^{h}_{C}\ V^{r}_{(a)})) \Big] - B \mid C = 0$$

and

$$\Big[{}^{*}C^{(p)D}_{AB} \widetilde{\Omega}_{(b)DC} - B^{jkh}_{ABC}\ N_{(b)l}\ C^{(p)l}_{jk||h} - C^{(p)i}_{jk}\ v^{j}_{AC}\ N_{(b)l}\ B^{k}_{B}$$

$$+ C^{l}_{rh|j}\ y^{j}\ N_{(b)l}\ B^{h}_{C}\ ({}^{*}C^{(p)D}_{AB}\ B^{r}_{D} + \sum_{a} \Omega_{(a)AB}\ V^{r}_{(a)})$$

$$+ \sum_{a} \Omega_{(a)AB}\ Z_{(ab)C} + \Omega_{(a)AB|C}\ d_{(ab)}) \Big] - B \mid C = 0 \qquad \ldots(5.13)$$

where

$$^{*}C^{(p)D}_{AB} := C^{(p)D}_{AB} - Q^{D}_{AB}\ .$$

Equations (5.12) and (5.13) are the generalised Gauss-Codazzi equations of fourth kind.

6. Special Cases

Case I—For $m = n - 1$, all these equations corresponding to Gauss-Codazzi, reduce to a case of a hypersurface of a Finsler manifold.

Case II—When the given congruence is formed of tangents to a one parameter family of curves such that $V^i_{(a)} = t^A_{(a)} \; B^i_A$, then we have

$$d_{(ab)} = 0, \; Q^E_{AB} = \sum_a t^E_{(a)} \; \Omega_{(a)AB}, \; X^D_{(a)B} = t^D_{(a);B}, \; \coprod^D_{(a)B} = t^D_{(a),B}$$

$$Y_{(ab)B} = t^A_{(a)} \; M_{(b)AB}, \; Z_{(ab)B} = t^D_{(a)} \; \widetilde{\Omega}_{(b)DB}.$$

In this case equations (4.11), (4.12) (5.7), (5.8), (5.9) and (5.10) respectively reduce to

$$S^{(p)}_{AFBC} = S^{(p)}_{rhjk} \; B^{rhjk}_{AFBC} + \tfrac{1}{2} \Big(\sum_a M_{(a)AC} \; M_{(a)BF} \Big) - B \mid C \qquad \ldots(6.1)$$

$$S^{(p)}_{rhjk} \; B^{rjk}_{ABC} \; N^h_{(a)} + \tfrac{1}{2} \Big[M_{(j)DB} \; C^{(p)D}_{AC} + \sum_b M_{(b)AC} \; M^{(p)}_{(ab)B} \Big] B|C=0 \qquad \ldots(6.2)$$

$$\Big[P^{(p)E}_{BCA} + C^{(p)E}_{CD} \; P^D_{BA} + B^{jkh}_{ABC} \; B^E_i \; (P^{(p)i}_{hkj} + P^i_{hj} \; C^{(p)l}_{kl}) + \sum_a M^{(p)E}_{(a)B} \; \widetilde{\Omega}_{(a)AC} \Big] - B \mid C = 0 \qquad \ldots(6.3)$$

$$\Big[B^{jkh}_{ABC} \; N_{(b)l} \; (P^{(p)l}_{hkj} + P^i_{hj} \; C^{(p)l}_{ki}) + C^{(p)D}_{AC} \; \widetilde{\Omega}_{(b)DB} + \sum_a M^{(p)}_{(ab)B} \; \widetilde{\Omega}_{(a)AC} \Big] - B \mid C = 0 \qquad \ldots(6.4)$$

$$\Big[{}^*P^{(p)D}_{BCA} + B^{jkh}_{ABC} \; B^E_i \; (P^{(p)l}_{hkj} + C^{(p)l}_{kl} \; P^i_{hj}) + \Lambda^E_{DC} \; C^{(p)D}_{AB} - C^{(p)E}_{DB} \; \Lambda^D_{AC} + \sum_a M^{(p)E}_{(a)B} \; \widetilde{\Omega}_{(a)AC} \Big] - B \mid C = 0 \qquad \ldots(6.5)$$

and

$$\Big[B^{jkh}_{ABC} \; N_{(b)l} \; (P^{(p)l}_{hkj} + C^{(p)l}_{kl} \; P^i_{hj}) = \widetilde{\Omega}_{(b)DB} \; C^{(p)D}_{AC} + M^{(p)}_{(b)DC} \; \Lambda^D_{AB} + \sum_a M^{(p)}_{(ab)B} \; \widetilde{\Omega}_{(a)AC} \Big] - B \mid C = 0. \qquad \ldots(6.6)$$

Case III—When the given congruence is a normal congruence we have

$$t^{A}_{(a)} = 0,\ d_{(ab)} = \delta_{ab},\ V^{i}_{(a)} = N^{i}_{(a)},\ \omega^{i}_{AB} = 0,\ X^{D}_{(a)B}$$

$$= -M^{D}_{(a)B},\ Y_{(ab)} = ((p-2)/2L)\, l_B\, \delta_{ab},\ L^{i}_{AB} = U^{i}_{AB},\ \Omega_{(b)AB}$$

$$= M_{(a)AB},\ \prod{}^{D}_{(a)B} = -\Omega_{(a)CB}\, g^{CD} - E_{lhk}\, g^{AD}\, B^{lk}_{AB}\, N^{h}_{(a)},\ Z_{(ab)B}$$

$$= N_{(ab)B}\ \ Q^{D}_{AB} = 0 \text{ and } {}^{*}C^{(p)F}_{AB} = C^{(p)F}_{AB}.$$

In this case (4.11), (4-12), (5.7), (5,8), (5.9) and (5.10) respectively reduce to

$$S^{(p)}_{AFBC} = S^{(p)}_{rhjk}\ B^{rhjk}_{AFBC} \qquad \ldots(6.7)$$

$$S^{(p)}_{rhjk}\ B^{rjk}_{ABC}\ N^{h}_{(c)} - \tfrac{1}{2}\Big[M_{(c)DC}\, C^{(p)D}_{AB} + M_{(c)AB;C} + M_{(c)AB}\, l_C\, ((p-2)/2L) - \sum_a M_{(a)AB}\, M^{(p)}_{(ac)C} \Big] - B \mid C = 0. \qquad \ldots(6.8)$$

$$\Big[P^{(p)E}_{BCA} + C^{(p)E}_{CE} P^{D}_{BA} + B^{jkh}_{ABC}\ B^{E}_{i}\ (P^{(p)i}_{hkj} + P^{i}_{hj}\ C^{(p)l}_{kl}) + \sum_a (M^{(p)E}_{(a)B}\ \tilde{\Omega}_{(a)AC} - M_{(a)AB} \prod{}^{E}_{(a)C}) \Big] - B \mid C = 0 \qquad \ldots(6.9)$$

$$\Big[B^{jkh}_{ABC}\ N_{(b)l}\ (P^{(p)l}_{hk} + C^{(p)i}_{kl} P^{l}_{hi} + \tilde{\Omega}_{(b)DB}\ C^{(p)D}_{AC} - M_{(b)AB;C} + \sum_a (M^{(p)}_{(ab)B}\ \tilde{\Omega}_{(a)AC} - Z_{(ab)C}\, M_{(a)AB}) \Big] - B \mid C = 0 \qquad \ldots(6.10)$$

$$\Big[{}^{*}P^{(p)D}_{BCA} + C^{(p)D}_{AB}\ \Lambda^{E}_{DB} - C^{(p)E}_{DB}\ \Lambda^{D}_{AC} + B^{jkh}_{ABC}\ B^{E}_{i}\ (P^{(p)i}_{hkj} + C^{(p)l}_{kl} P^{i}_{hj}) + \sum_a (M^{(p)E}_{(a)B}\ \tilde{\Omega}_{(a)AC} - M_{(a)AB} \prod{}^{E}_{(a)C}) \Big] - B \mid C = 0 \qquad \ldots(6.11)$$

and

$$\Big[(P^{(p)l}_{hkj} + C^{(p)l}_{kl}\ P^{i}_{hj})\ B^{jkh}_{ABC}\ N_{(c)l} + \tilde{\Omega}_{(c)DB}\ C^{(p)D}_{AC} - M_{(c)AB;C} - M^{(p)}_{(c)DB}\ \Lambda^{D}_{AC} + \sum_a (M^{(p)}_{(ac)B}\ \tilde{\Omega}_{(a)AC} - Z_{(a)C}\, M_{(ac)AB}) \Big] - B \mid C = 0. \qquad \ldots(6.12)$$

Case IV—For an affinely connected space equations (5.12) and (5.13) reduce to eqns. (5.7) and (5.8) respectively.

Case V—When F^n is a Minkowskian manifold $P^{l}_{jkh} = 0$, $K_{ijkh} = 0$, $C^{l}_{hk|r} = 0$, therefore we also have $R^{(p)r}_{ijk} = 0$ and $P^{(p)k}_{hij} = 0$. Tuus if both F^m and F^n are Minkowskian manifolds, eqns. (5.7), (5.8), (5.9) and (5.10) respectively reduce to

$$\left[Q^{E}_{AB,C} + \sum_{a} (M^{(p)E}_{(a)B} \widetilde{\Omega}_{(a)AC} - t^{E}_{(a)} \Omega_{(a)AB,C} - \Omega_{(a)AB} \prod^{E}_{(a)C}) \right] - B \mid C = 0 \qquad ...(6,13)$$

$$\left[{}^{*}C^{(p)D}_{AC} \widetilde{\Omega}_{(b)DB} + \sum_{a} (M^{(p)}_{(ab)B} \widetilde{\Omega}_{(a)AC} - d_{(ab)} \Omega_{(a)AB,C} - Z_{(ab)C} \Omega_{(a)AB}) \right] - B \mid C = 0 \qquad ...(6.14)$$

$$\left[C^{(p)E}_{AB,D} \Lambda^{D}_{CF} v^{F} + Q^{E}_{AB,C} + \Lambda^{E}_{DC} {}^{*}C^{(p)D}_{AB} - \sum_{a} (t^{E}_{(a)} \Omega_{(a)AB|C} + \Omega_{(a)AB} \prod^{E}_{(a)C} - M^{(p)E}_{(a)B} \widetilde{\Omega}_{(a)AC}) \right] - B \mid C = 0 \qquad ...(6.15)$$

and

$$\left[{}^{*}C^{(p)D}_{AC} \widetilde{\Omega}_{(b)DB} - M^{(p)E}_{(b)DB} \Lambda^{D}_{AC} + \sum_{a} (M^{(p)}_{(ab)} \widetilde{\Omega}_{(a)AC} - d_{(ab)} \Omega_{(a)AB|C} - Z_{(ab)C} \Omega_{(a)AB}) \right] - B \mid C = 0. \qquad ...(6.16)$$

Case VI—For $p = 2$, these Gauss-Codazzi equations will respectively reduce to Gauss-Codazzi equations obtained for $C\Gamma$-connection.

7. Some Curvature Properties

Corresponding to the curvature tensor $S^{(p)}_{ijhk}$ we here define a scalar $S^{(p)}(x, \zeta, X)$ as follows :

$$S^{(p)}(x, \zeta, X) : = S^{(p)}_{ijhk} \zeta^{i} \zeta^{h} X^{j} X^{k} / (g_{ih} g_{jk} - g_{ij} g_{hk}) \zeta^{i} \zeta^{h} X^{j} X^{k}. \qquad ...(7.1)$$

If $\zeta^{i} \eta_{A} B^{i}_{A} = X^{i} = Y_{A} B^{i}_{A}$, then we can define

$$(S^{(p)}_{(a)}(x, \zeta))^{-1} := M_{(a)AB}\, \eta^A\, \eta^B \qquad \ldots(7.2)$$

and

$$(*S^{(p)}_{(a)}(x, \zeta))^{-1} := M_{(a)AB}\, Y^A\, Y^B. \qquad \ldots(7.3)$$

Multiplying (4.11) by $Y^A\, Y^D\, \eta^C\, \eta^F$, we obtain

$$\begin{aligned} S^{(p)}_{AFBC}\quad & Y^A\, Y^B\, \eta_C\, \eta^F \\ &= S^{(p)}_{rhjk}\, X^r\, X^j\, \zeta^h\, \zeta^k + \tfrac{1}{2}\Big[g_{DF}\,(Q^D_{AB;C} - Q^D_{AC;B}) \\ &+ \sum_a \{M_{(a)AC} \times M_{(a)BF} - t_{(a)F}\,(\Omega_{(a)AB;C}) - \Omega_{(a)AC;B} + \Omega_{(a)AC}\, X_{(a)BF}\}\Big] \\ &\times Y^A\, Y^B\, \eta^C\, \eta_i^F + \tfrac{1}{2}\sum_a (\Omega_{(a)}\, X_{(a)} - (S^{(p)}_{(a)}\quad *S^{(p)}_{(a)})^{-1}) \qquad (7.4) \end{aligned}$$

where

$$\Omega_{(a)} = \Omega_{(a)AB}\, Y^A\, Y^B,\ X_{(a)} = X_{(a)AB}\eta^A\, \eta_i^B.$$

Dividing equation (7.4) by $(g_{ih}\, g_{jk} - g_{ij}\, g_{hk})\, \zeta^i\, \zeta^h\, X^j\, X^k = (g_{FC}\, g_{BA} - g_{FB}\, g_{CA})\, Y^A\, Y^B\, \eta_i^C\, \eta^F = 1 - (g_{AB}\, \eta^A\, Y^B)^2$ and using a relation similar to (7.1) for the scalar curvature of F^m we obtain on simplification

$$\begin{aligned} S^{(p)}(u, \eta_i, Y) = S^{(p)}(x, \zeta, X) &+ \tfrac{1}{2}\,(1 - (g_{AB}\eta^A\, Y^B)^2)^{-1} \\ &\cdot \Big[\sum_a (\Omega_{(a)}\, X_{(a)} - (S^{(p)}_{(a)} \cdot *S^{(p)}_{(a)})^{-1} \\ &+ \{g_{DF}\,(Q^D_{AB;C} - Q^D_{AC;B}) + \sum_a (M_{(a)AC}\, M_{(a)BF} \\ &- t_{(a)F}\,(\Omega_{(a)AB;C} - \Omega_{(a)AC;B}) + \Omega_{(a)AC}\, X_{(a)BF})\} \\ &\times Y^A\, Y^B\, \eta_i^C\, \eta^F] \qquad \ldots(7.5) \end{aligned}$$

which is a relation between the scalar curvatures of the submanifold F^m and the enveloping manifold F^n.

For case II of previous section this reduces to

$$\begin{aligned} S^{(p)}(u.\ \eta_i, Y) = S^{(p)}(x, \zeta, X) &- \tfrac{1}{2}\,(1 - (g_{AB}\, \eta^A\, Y^B)^2)^{-1} \\ &\sum_a \Big[(S^{(p)}_{(a}) \cdot *S^{(p)}_{(a)})^{-1} - M_{(a)AC}\, M_{(a)BF}\, Y^A\, Y^B\, \eta^C\, \eta^F \Big] \\ & \qquad \ldots(7.6) \end{aligned}$$

For a normal congruence (7.5) reduces to

$$S^{(p)}(u, \eta, Y) = S^{(p)}(x, \zeta, X). \qquad \ldots(7.7)$$

Remark : Similar results can also be obtained for other curvature tensors also.

8. Ricci Tensors

Multiplying eqn. (4.11) by g^{EF} and using

$$g^{EF} B^i_F = g^{ij} B^E_j , \; g^{EF} S^{(p)}_{AFBC} = S^{(p)E}_{ABC} ,$$

we can obtain on simplificatiou

$$S^{(p)E}_{ABC} = S^{(p)l}_{rjk} B^{rjk}_{ABC} B^E_i + \tfrac{1}{2}\Big[Q^E_{AB;C} + \sum_a \Big\{ M_{(a)AC} M^E_{(a)B} - t^E_{(a)} \Omega_{(a)AB;C} - \Omega_{(a)AB} X^E_{(a)C} \Big\}\Big] - B \mid C. \qquad \ldots(8.1)$$

If $S^{(p)E}_{ABE} = S^{(p)}_{AB}$, $S^{(p)l}_{rjl} = S^{(p)}_{rj}$, then contracting equation (8.1) for E and C we obtain

$$S^{(p)}_{AB} = B^{rj}_{AB} (S^{(p)}_{rj} - \sum_a S^{(p)l}_{rjk} N_{(a)l} N^k_{(a)}) + \tfrac{1}{2} \Big[Q^E_{AB;E} - Q^E_{AE;B}) \sum_a \Big\{ M_{(a)AE} M^E_{(a)B} - M_{(a)AB} M^E_{(a)E} - t^E_{(a)} (\Omega_{(a)AB;E} - \Omega_{(a)AE;B}) - \Omega_{(a)AB} X^E_{(a)E} + \Omega_{(a)AE} X^E_{(a)B} \Big\}\Big] . \qquad \ldots(8.2)$$

which is a relation between the Ricci tensors of F^n and F^m.

For a one parameter family of curves equation (8.2) reduces to

$$S^{(p)}_{AB} = S^{(p)}_{rj} B^{rj}_{AB} - \sum_a S^{(p)l}_{rjk} N_{(a)l} N^k_{(a)} B^{rj}_{AB} + \tfrac{1}{2} \sum_a M_{(a)}{}^{AE} M^E_{(a)B} - M_{(a)}{}^{AB} M^E_{(a)E} . \qquad \ldots(8.3)$$

For a normal congruence equation (8.2) reduces to

$$S^{(p)}_{AB} = B^{rj}_{AB} (S^{(p)}_{rj} \sum_a S^{(p)l}_{rjk} N_{(a)l} N^k_{(a)}). \qquad \ldots(8.4)$$

From (1.12) we can obtain

$$S^{(p)}_{ij} = S_{ij} - L^{-2} h_{ij} (n - 2) (p - 2)^2/4 (p - 1) \qquad \ldots(8.5)$$

which gives for $S^{(p)}_{ij}\, g^{ij} = S^{(p)}$ etc.

$$S^{(p)} = S - L^{-2}\,(n-1)\,(n-2)\,(p-2)^2/4(p-1). \qquad \ldots(8.6)$$

If we substitute from (8.5) and (8.6) in (8.2) we can obtain a relationship between the Riccci tensors and curvature scalars based on S^{r}_{ijk} in F^n and its counterpart in F^m.

Remark : Following similar methods we can also oblain the relationship between the Ricci tensors and curvature scalars based on other curvature tensors.

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