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WAVES DUE TO A MOVING OSCILLATORY SURFACE PRESSURE IN A LAYERED FLUID OF FINITE DEPTH

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The two-dimensional initial-value problem of waves due to an oscillatory pressure distribution moving uniformly on the surface of a two-layered fluid of finite depth is solved by the integral transform method. Asymptotic analysis of the infinite integrals for both the surface and the interfacial waves are given for large distances and times. It is noted that the stratification sharply changes the number and the character of the progressive waves and also introduces two critical speeds instead of one as found for a homogeneous fluid. At these speeds the solution becomes singular.

1. Introduction

The problem of waves generated by a moving oscillatory pressure distribution on the surface of a homogeneous fluid of infinite depth has been analysed by Kaplan[1] and the analysis has been extended to a two-layered fluid by Pramanik[2]. On the other hand Debnath and Rosenblat[3] investigated the same problem for a homogeneous fluid of of finite depth.

In the present paper, we consider the two-dimensional initial-value problem of waves due to an oscillatory pressure moving uniformly on the surface of a fluid of depth d and density ρ_2, overlying another fluid of depth h and density ρ_1. The integral representations of the displacements of both the free surface and the interface are obtained with the help of the integral transform method. The chief hurdle in the subsequent asymptotic analysis arises from the fact that the wave-frequencies, given by the equation

$$A\sigma^4 - B\sigma^2 + C = 0$$

as functions of the wave number k are too complicated to allow use of the usual methods. However, two important results due to Wehausen[4] and Mahanti[5,6] permit us to express the two frequencies in the form of homogeneous fluid-like $\sigma - k$ relations

$$\sigma_1^2 = gk \tanh k(h - h_0), \quad \sigma_2^2 = gk \tanh k h_0.$$

where h_0 $(0 < h_0 < d)$ is indepenndent of k. Assuming next the prescribed pressure as a generalized function, we employ a theorem due to Lighthill[7] to obtain the steady-state solution of the problem at large distances. It is noted that the presence of two layers introduces two critical speeds instead of one as found in homogeneous fluids. Depending upon the relative magnitudes of the two critical speeds and the speed of the pressure strip, the total number of ultimate progressive waves on the frontside and on the backside taken together on either surface turns out to be eight, six or four.

2. Formulation of the Problem

A heavy homogeneous fluid of density ρ_2 and of uniform depth d lies on another heavy homogeneous fluid of density ρ_1 ($> \rho_2$) and of uniform depth $h - d$, both fluids being of infinite horizontal extent, initially both the fluids are at rest and the surface of separation is a horizontal plane. Waves are generated both on the free surface and on the interface by the action of a pressure distribution $f(x) \exp(i\omega t)$ which is suddenly applied on the free surface at the initial moment and is then made to move continually with an uniform velocity V. For the two-dimensional motion in the xy-plane, a coordinate system is chosen with origin on the undisturbed free surface, where y-axis is positive upwards and the x-axis positive to the right. It is convenient to pose the problem in a moving coordinate system in which the origin moves with the velocity v along the positive x-direction, so that the applied pressure strip is fixed with respect to this system. Let $y = \eta(x, t)$ and $y = -d + \zeta(x, t)$ be the equations to the free surface and the surface of separation respectively at any subsequent time t. With usual notations, the linearized equations of wave motion and the other necessary conditions are

(i) $\nabla_1^2 \varphi_j = 0,\ x \in (-\infty, \infty),\ y \in [-d, 0]$

and $y \in [-h, -d]$ for $j = 2, 1$ respectively. ...(1)

(ii) $\rho_2^{-1} f(x) \exp(i\omega t) + \varphi_{2t} - v\varphi_{2x} + g\eta = 0,\ -d < y \leqslant 0, t > 0$ ...(2)

$\eta_t - V\eta_x = \varphi_{2y}$, on $y = \eta = 0,\ t > 0$ ...(3)

(iii) $\rho_2(\varphi_{2tt} - 2V\varphi_{2tx} + V^2\varphi_{2xx}\, g\varphi_{2y}) = \rho_1(\varphi_{1tt} - 2V\varphi_{1tx} + V^2\varphi_{1xx}$
$+ g\varphi_{1y})$, on $y = -d$. ...(4)

$\varphi_{2y} = \varphi_{1y},\ y = -d$ ...(5)

(iv) $\varphi_j = 0,\ \eta = \zeta = 0$, for $t = 0$ ...(6)

(v) $\varphi_{jx} = 0,\ \varphi_j = 0,\ 1x \to \infty\ (j = 2,1)$ ...(7)

(vi) $\varphi_{1y} = 0$ on $y = -h$. ...(8)

The suffixes 2 and 1 refer to upper and lower fluids respectively.

3. Solution

The non-temporal part of the applied pressure $f(x)$ is assumed to be a generalised function. The Fourier transform of φ with respect to x is denoted by $\bar{\varphi}$:

$$\bar{\varphi}(k, y; t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \varphi(x, y; t) \exp(-ikx)\, dx. \qquad \ldots(9)$$

Application of the usual Fourier transform method followed by a slight simplification enables us to write the following solution of the system (1)-(8):

$$(2\pi)^{1/2} g\eta = \int_0^{\infty} [I(k) + I(-k)]\, dk \qquad \ldots(10)$$

$$(2\pi)^{1/2} g\zeta = \int_0^{\infty} [J(k) + J(-k)]\, dk \qquad \ldots(11)$$

where

$$I(k) = \left\{2\rho_2\left(\sigma_1^2 - \sigma_2^2\right)\right\}^{-1} \bar{f}(k)(\omega - kv)\exp(ikx)$$

$$\times\, [\,\{(\omega - vk)^2 - (\sigma_1\sigma_2)^2 (gk)^{-1} \coth kd\}$$

$$\times \sum_{l=1}^{2}\sum_{m=1}^{2} (-1)^{l+m}\{\sigma_l - (-1)^{m-1}(\omega - kv)\}^{-1} \exp i\{ikv + (-1)^{m-1}$$

$$\times\, i\sigma_l\}\, t] + \sum_{l=1}^{2}\sum_{m=1}^{2}(-1)^{l+m}\{\sigma_l + (-1)^{m-1}(\omega - kv)\} \exp i\{kv$$

$$+ (-1)^{m-1}\sigma_l\}\, t] - \left\{\rho_2\left(\sigma_1^2 - \sigma_2^2\right)\right\}^{-1} \bar{f}(k)(\omega - kv)^2\ \{(\omega - kv)^2$$

$$- (\sigma_1\sigma_2)^2 (gk)^{-1} \coth kd\} \sum_{m=1}^{2} (-1)^{m-1}\left\{\sigma_m^2 - (\omega - kv)^2\right\}^{-1}$$

$$\times \exp\{i(\omega t + kx)\} - \left\{2\rho_2\left(\sigma_1^2 - \sigma_2^2\right)\right\}^{-1} \bar{f}(k)\, gk \exp i(ikx)$$

$$\times\, [(\tanh k h_0 - \tanh kd) \sum_{m=1}^{2} \exp\{ikv + (-1)^{m-1} i\sigma_1\}\, t$$

$$-\{\tanh k(h - h_0) - \tanh kd\} \sum_{m=1}^{2} (-1)^{m-1} \exp i\{kv$$

$$+ (-1)^{m-1}\sigma_2\}\, t] - \rho_2^{-1}\bar{f}(k) \exp\{i(\omega t + kx)\}; \qquad \ldots(12)$$

and

$$J(k) = \left\{2\rho_2(\rho_1-\rho_2)\left(\sigma_1^2-\sigma_2^2\right)\right\}^{-1}\bar{f}(k)(\omega-kv)$$

$$\times [\text{sech}\, k(h-h_0)\{\rho_1 \coth k(h-d)\sinh k(h-h_0-d)$$

$$-\rho_2\cosh k(h-h_0-d)\}][\{(\omega-kv)^2-(\sigma_1\sigma_2)^2(gk)^{-1}\coth kd\}$$

$$\times \sum_{l=1}^{2}\sum_{m=1}^{2}(-1)^{l+m}\{\sigma_l-(-1)^{m-1}(\omega-kv)\}^{-1}\exp i\{kv$$

$$+(-1)^{m-1}\sigma_l\}t+\sum_{l=1}^{2}\sum_{m=1}^{2}(-1)^{l+m}\{\sigma_l+(-1)^{m-1}(\omega-kv)\}$$

$$\times \exp i\{kv+(-1)^{m-1}\sigma_l\}t]\exp(ikx)-\{\rho_2(\rho_1-\rho_2)gk$$

$$\times\left(\sigma_1^2-\sigma_2^2\right)\}^{-1}(\omega-kv)^2\bar{f}(k)\{(\omega-kv)^2-(\sigma_1\sigma_2)^2(gk)^{-1}$$

$$\times\coth kd\}\exp\{i(\omega t+kx)\}[(\omega-kv)^2\{\rho_1\coth k(h-d)$$

$$\cosh kd+\rho_2\sinh kd\}-gk(\rho_1\coth k(h-d)\sinh kd$$

$$+\rho_2\cosh kd\}]\sum_{m=1}^{2}(-1)^{m-1}\left\{\sigma_m^2-(\omega-kv)^2\right\}^{-1}-2\rho_2(\rho_1-\rho_2)$$

$$(\sigma_1-\sigma_2\}^{-1}\bar{f}(k)\,gk\;[\text{sech}\, k(h-h_0)\{\rho_1\coth k(h-d)\sinh k$$

$$(h-h_0-d)-\rho_2\cosh k(h-h_0-d)\}]\;[(\tanh kh_0$$

$$-\tanh kd)\sum_{m=1}^{2}(-1)^{m-1}\exp i\{kv+[-1)^{m-1}\sigma_1\}t]$$

$$-\{\tanh k(h-h_0)-\tanh kd\}\sum_{m=1}^{2}\exp i\{kv+(-1)^{m-1}\sigma_2\}t]\exp(ikx)$$

$$-\{\rho_2(\rho_1-\rho_2)gk\}^{-1}\bar{f}(k)(\omega-kv)^2$$

$$\times\{\rho_1\coth k(h-d)\cosh kd+\rho_2\sinh kd\}$$

$$\times\exp\{i(\omega t+kx)\}. \qquad \ldots(13)$$

The quantities $\pm\sigma_1$, $\pm\sigma_2$ are the roots of the frequency equation

$$\sigma^4-gk\rho_1\{\rho_1+\rho_2\tanh kd.\tanh k(h-d)\}^{-1}\{\tanh kd+\tanh k(h-d)\}\sigma^2$$

$$+(\rho_1-\rho_2)(gk)^2\{\rho_1+\rho_2\tanh kd.\tanh k(h-d)\}^{-1}\tanh kd.$$

$$\tanh k(h-d)=0. \qquad \ldots(14)$$

As already stated[1], there exists a constant h_0 $(0 < h_0 < d)$, such that

$$\sigma_1^2 = gk\tanh k(h-h_0),\ \ \sigma_2^2 = gk\tanh kh_0.$$

4. Asymptotic Analysis of η and ζ in the Steady-state ($t \to \infty$)

The dominant parts of the asymptotic values of η and ζ in the steady-state are determined by using the following result due to Lighthill[7] :

if $f(k)$ has a simple pole at $k = \alpha$; then as $|x| \to \infty$,

$$\int_a^b f(k)\, e^{ikx}\, dk = \pi\, i \operatorname{sgn} x \,(\text{Residue of } f(k)\, e^{ikx} \text{ at } k = \alpha) + O(1/|x|),$$

$$\text{when } a < \alpha < b. \qquad \ldots(15)$$

The real positive poles of the integrands in either of the integrals for η and ζ are the zeros of $\sigma_j - \omega - kv$, $\sigma_j + \omega - kv$ and $\sigma_j - \omega + kv$. For $j = 1$, these zeros are designated as (α_1, α_2), α_3 and α_4 respectively. While for $j = 2$, the corresponding roots are denoted by (β_1, β_2), β_3 and β_4 respectively. For $j = 1$, we see from Fig. 1 that α_3 and α_4 both exist, and are distinct, for all ω, v, h while α_1, α_2 are distinct, coincident or unreal according as $v \lesseqgtr V_1^*$ The value V_1^* of V is given by the condition of tangency of the line $Y = kV + \omega$ to the curve $Y = \sigma_1$, namely the k-eliminant of the equations

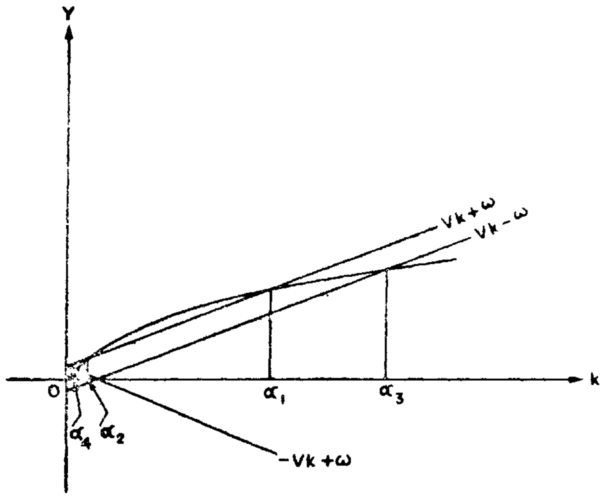

Fig. 1. Positions of the poles α_n given by the intersection of the curve $Y = \sigma_1$ with the lines $Y = Vk + \omega$, $Y = Vk - \omega$ and $Y = -Vk + \omega$.

$$\sigma'_1(k) = v,\ \sigma_1(k)\, kv + \omega. \qquad \ldots(16)$$

This case $v = v_1^*$, called the critical case, is discussed later on. Similar statements hold in respect of the roots β_j ($j = 1$ to 4), the critical velocity v_2^* in this case being given by the conditions

$$\sigma_2' (k) = v, \; \sigma_2 (k) = kv + \omega. \qquad \ldots(17)$$

For $v \neq v_j^*$ $(j = 1, 2)$, the various integrals occurring in $I(\pm k)$ and $J(\pm k)$ are evaluated by (15) for $t \to \infty$ or $|x| \to \infty$ as the case may be.

The final asymptotics for η and ζ are as follows :

$$(2/\pi)^{1/2} ig\rho_2 \eta \simeq - H\left(v_1^* - v \right) \psi_1 (- \alpha_2) \bar{f}(- \alpha_2) \exp \{i (\omega t - \alpha_2 x)\}$$

$$+ H\left(v_2^* - v\right) \psi_2 (- \beta_2) \bar{f}(- \beta_2) \exp \{i \omega t - \beta_2 x)\},$$

$$x \to \infty;$$

$$(2/\pi)^{1/2} ig\rho_2 \eta \simeq H \left(v_1^* - v \right) \psi_1 (- \alpha_1) \bar{f} (- \alpha_1) \exp \{i (\omega t - \alpha_1 x)\}$$

$$- \psi_1 (\alpha_3) \bar{f}(\alpha_3) \exp \{i (\omega t + \alpha_3 x)\}$$

$$+ \chi_1 (\alpha_4) \bar{f}(\alpha_4) \exp \{i (\omega t + \alpha_4 x)\}$$

$$- H \left(v_2^* - v\right) \psi_2 (- \beta_1) \bar{f} (- \beta_1) \exp \{i (\omega t - \beta_1 x)\}$$

$$+ \psi_2 (\beta_3) \bar{f}(\beta_3) \exp \{i (\omega t + \beta_3 x)\}$$

$$- x_2 (\beta_4) \bar{f}(\beta_4) \exp \{i (\omega t + \beta_4 x)\} \; x \to - \infty \qquad \ldots(18)$$

where

$$\psi_j (k) = \left[(\omega - kv)^2 - \frac{gk (1 - \epsilon) \tanh (h - d)}{1 + \epsilon \tanh kd. \tanh k (h - d)} \right] \times \frac{(kv - \omega)}{(\sigma_1^2 - \sigma_2^2) (\sigma_j' - v)}$$

$$x_j (k) = \left[(\omega - kv)^2 - \frac{gk (1 - \epsilon) \tanh k (h - d)}{1 + \epsilon \tanh kd. \tanh k (h - d)} \right] \times \frac{(kv - \omega)}{(\sigma_1^2 - \sigma_2^2) (\sigma_j' + v)}$$

$$\epsilon = \rho_2/\rho_1, \; j = 1, 2.$$

And

$$(2/\pi)^{1/2} ig (1 - \epsilon) \rho_2 \zeta \simeq - H (v_1^* - v) M_1 (-\alpha_2) \psi_1 (- \alpha_2) \bar{f} (- \alpha_2) \times \exp \{i (\omega t - \alpha_2 x)\}$$

(equation continued on p. 959)

$$+ H\left(v_2^* - v\right) M_2(-\beta_2)\,\psi_2(-\beta_2)\,\bar{f}(-\beta_2)\exp\{i\,\omega t - \beta_2 x)\}$$
$$x \to \infty;$$

$$(2/\pi)^{1/2}\, ig\,(1-\epsilon)\,\rho_2\,\zeta \cong - M_1(\alpha_3)\,\psi_1(\alpha_3)\,\bar{f}(\alpha_3)\exp\{i\,\omega t + \alpha_3 x)\}$$
$$+ M_1(\alpha_4)\,\chi_1(\alpha_4)\,\bar{f}(\alpha_4)\exp\{i\,(\omega t + \alpha_4 x)\}$$
$$+ H\left(v_1^* - v\right) M_1(-\alpha_1)\,\psi_1(-\alpha_1)\,\bar{f}(-\alpha_1)\exp\{i\,(\omega t - \alpha_1 x)\}$$
$$+ H\left(v_2^* - v\right) M_2(-\beta_1)\,\psi_2(-\beta_1)\,\bar{f}(-\beta_1)\exp\{i\,(\omega t - \beta_1 x)\}$$
$$+ M_2(\beta_3)\,\psi_2(\beta_3)\,\bar{f}(\beta_3)\exp\{i\,(\omega t + \beta_3 x)\}$$
$$- M_2(\beta_4)\,\chi_2(\beta_4)\,\bar{f}(\beta_4)\exp\{i\,(\omega t + \beta_4 x)\},\quad x \to -\infty \qquad ...(19)$$

where

$$M_1(k) = \frac{\sinh k\,(h-h_0-d) + (1-\epsilon)\sinh k\,(h-d).\,\cosh k\,(h-h_0-d)}{\sinh k\,(h-h_0).\,\cosh k\,[h-h_0)}$$

$$M_2(k) = \frac{\sinh k\,(h-h_0) + (1-\epsilon)\sinh k\,(h-d).\,\cosh k\,(h-d)}{\cosh kh_0\ \ \sinh k\,(h-d)}.$$

5. Critical Case

When $v = v_1^*$, let $\alpha_1 = \alpha_2 = \alpha_0$ (say).

It is easily seen that $\eta\,(x, t)$ and $\zeta\,(x, t)$ both become singular on the critical curves. For exampe, of the four integrals in η wherein poles are the zeros of $\sigma_j - \omega - kv$, the two proportional to $\exp\,(i\omega t)$ remain fiinte as $t \to \infty$, $|\,x\,| \to \infty$; the other two are of the form

$$\int_0^\infty F(k)\,(\sigma_j - \omega - kv)^{-1}\exp\{-\,ikx + i\,(-kv + \sigma_j)\,t\}\,dk. \qquad ...(20)$$

For $j = 1$, the above is asymptotically equivalent to

$$2^{1/2}\,(-3/2)!\;t^{1/2}\,[\sigma''(\alpha_0)]^{-1}\,F(\alpha_0)\exp\{i\,\pi/4 + i\,(\omega t - \alpha_0\,x)\},$$
$$t \to \infty,\ v = v_1^* \qquad ...(21)$$

The last expression clearly becomes infinite as $t \to \infty$.

6. Physical Discussion

I. *General*

When $v \neq v_j^*$ $(j = 1, 2)$, we see from (18) and (19) that the steady-state wave system consists of several simple progressive waves. The numbers and distributions of the surface and interfacial waves relative to the pressure system follow the same rule and so we mention the characteristics of the surface waves only. From (18), we see that there are three possible distributions of surface waves.

(i) If ν is less than both ν_1^* and ν_2^*, the total number of the waves is eight, two on the frontside and six on the back side.

(ii) If ν lies between ν_1^* and ν_2^* there are six waves, one on the frontside and five on backside.

(iii) If ν exceeds both ν_1^* and ν_2^* there are four waves, all of which are on the backside.

The waves on the frontside are of lengths 2π $(\alpha_2^{-1}, \beta_2^{-1})$, and they move with constant velocities ω $(\alpha_2^{-1}, \beta_2^{-1})$ along the positive x-direction. The waves on backside are of lengths 2π $(\alpha_1^{-1}, \beta_1^{-1}, \alpha_3^{-1}, \beta_3^{-1}, \alpha_4^{-1}, \beta_4^{-1})$, and these move with constant velocities ω $(\alpha_1^{-1}, \beta_1^{-1}, \alpha_3^{-1}, \beta_3^{-1}, \alpha_4^{-1}, \beta_4^{-1})$ the first two in the positive x-direction and the last four in the negative x-direction. The amplitudes of all these waves are independent of x and t; these however depend on the parameters

$$\nu' = \nu\,(gh^{-1})^{1/2},\ \omega' = \omega\,(h/g)^{1/2},\ d' = d/h,\ \epsilon = p_2/p_1.$$

It is of some interest to note that the particular wave distribution (out of the three mentioned above) which corresponds to ony given set of values of the parameters ν', ω', d', ϵ may be ascertained by simple geometry. For this, we introduce the dimensionless parameters

$$a = \omega\nu g^{-1},\ b = \nu\,\{g\,(h - h_0)\}^{-1/2}.$$

Equations (16) representing the critical case $(\alpha_1 = \alpha_2)$ may be replaced by the following equations

$$a = (1/4)\tanh\alpha\,(1 - \alpha^2\coth\alpha.\ \mathrm{sech}^4\,\alpha) \qquad \ldots(22)$$

$$b = (1/2)\,(\alpha^{-1}\tanh\alpha)^{1/2}\,(1 + \alpha\coth\alpha.\ \mathrm{sech}^2\,\alpha) \qquad \ldots(23)$$

where $\alpha = \alpha_0\,(h - h_0)$.

Similarly eqns. (17) representing the critical case $(\beta_1 = \beta_2)$ take the form

$$a = (1/2)\tanh\alpha\,(1 + \beta\coth\beta.\ \mathrm{sech}^2\,\beta) - (1/4)\tanh\beta\,(1 + \beta\coth\beta.\ \mathrm{sech}^2\,\beta) \qquad \ldots(24)$$

$$bc = (1/2)\,(\beta^{-1}\tanh\beta)^{1/2}\,(1 + \beta\coth\beta.\ \mathrm{sech}^2\,\beta) \qquad \ldots(25)$$

where

$$\beta_0 = \beta_1 = \beta_2, \beta_0 h_0 = \beta, \ c = \{h_0^{-1} (h - h_0)\}^{1/2}.$$

Equations (22) – (23) and (24) — (25) are the parametric equations of two curves (henceforth called the critical curves) AB and AD in the first quadrant of the (a, b)-plane (Fig. 2, $c > 1$; Fig. 3, $c < 1$); for points (a, b) an these curves, we have $v = v_1^*$ and $v = v_2^*$ respectively.

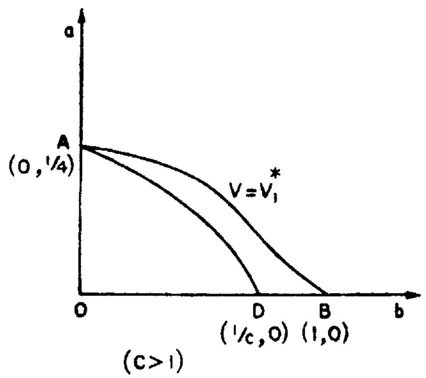

FIG. 2. Critical curves in $(a - b)$-plane.

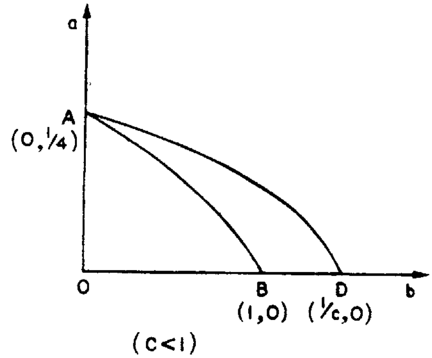

FIG. 3. Critical curves in $(a - b)$-plane.

When $c > 1$ (Fig. 2), as one moves along any radius vector OP (other than a- and b-axes) drawn from the origin towards the curves, it is found that v increases from

(i)′ the value zero to v_2^* on the segment of OP lying between 0 and the curve AD,

(ii)′ the value v_2^* to v_1^* on the segment of OP between the curves AD and AB.

(iii)′ the value v_1^* to ∞ on the segment of OP beyond the curve AB.

Thus the critical curves divide the first quadrant of the (a, b)-plane into three regions. Recalling the three types of wave distributions (i) – (iii), we see that

(1) in the first region bounded by the a-axis, the b-axis and the curve AD, $v < v_2^* < v_1^*$ and so there are eight vaves as described in (i).

(2) in the second region bounded by the b-axis and the curves AD, AB, $v_2^* < v < v_1^*$ and so there are six waves as described in (ii).

(3) in the third region which is the open region beyond the curve AB, $v > v_1^* > v_2^*$ and so there are four waves as described in (iii).

The corresponding statements for $c < 1$ (Fig. 3) are obtained on interchanging the symbols 'v_1^* ' and 'v_2^* ' as also 'AB' and 'AD' in (i)′ - (iii)′ and (1) - (3).

II. *An illustrative Case*

Let

$$\rho_0(x, t) = \frac{\rho}{2a_1} \cos \omega t, \ |x| < a_1$$
$$= 0, \qquad |x| > a_1.$$

Then

$$\bar{f}(k) = (2\pi)^{-1/2} \ \rho \{\sin (ka_1)/(ka_1)\}.$$

For $v > \left(v_1^*, v_2^* \right)$ eqn. (18) gives

$$\eta = 0, \ x \to \infty$$
$$2g\rho_2\eta/\rho = -\psi_1(\alpha_3)\{\sin(\alpha_3 a_1)/(a_1 \alpha_3)\} \sin(\alpha_2 x + \omega t)$$
$$+ \chi_1(\alpha_4)\{\sin(\alpha_4 a_1)/(a_1\alpha_4)\} \sin(\alpha_4 x + \omega t)$$
$$+ \psi_2(\beta_3)\{\sin(\beta_3 a_1)/(a_1\beta_3)\} \sin(\beta_3 x + \omega t)$$
$$- \chi_2(\beta_4)\{\sin(\beta_4 a_1)/(a_1\beta_4)\} \sin[\beta_4 x + \omega t), \quad x \to -\infty. \qquad \ldots(26)$$

The characteristics of this type of wave motion are approximately illustrated here graphically (Figs. 4, 5, 6, 7) by plotting (i) $g\rho_2 h\eta/\rho$ against x/h, $|x| \gg |$, for two different values of $v' = (gh)^{-1/2} v$, and (ii) the wave velocity $c' = \omega'/\alpha_3'$, $(\alpha_3' = h\alpha_3)$

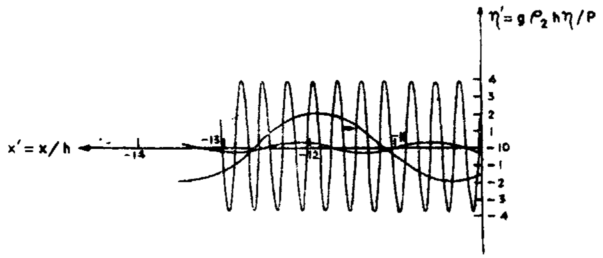

FIG. 4. Variation of η' with x', $h = 1$, $\epsilon = 0.5$, $d = 0.1$, $h_0 = 0.038$, $\omega' = 2$, $V' = 0.3$, $a_1 = 0.25$.

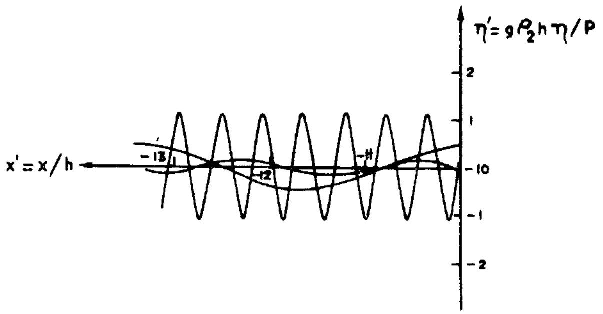

FIG. 5. Variation of η' with x'. $V' = 0.4$, other parameters are as under figure 4.

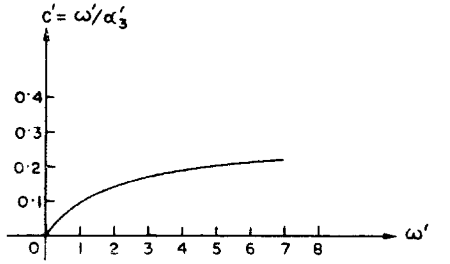

FIG. 6 Variation of c' with ω. $V' = 0.4$; Other parameters excluding a_1 are as under figuer 4.

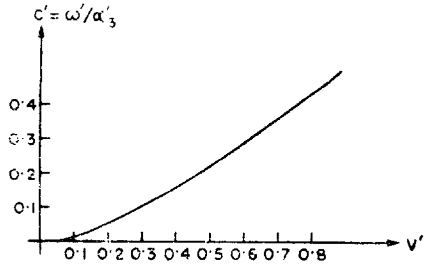

FIG. 7 Variation of c' with V', $\omega' = 3$; Other parameters excluding a_1 are as under figure 4.

against v' as well as $\omega' = \omega\,(h/g)^{1/2}$. The values of the parameters due to Mahanti[8] which are kept constant, as and when required, in (i) and (ii) are given below:

$$h = 1,\ \epsilon = 0.5,\ d = 0.1,\ h_0 = 0.038,$$

$$\omega' = 2,\ a_1 = 0.25;\ v' = 0.3 \text{ and } 0.4$$

for two different cases in (i) For variation of c' with v' we take $\omega' = 3$ in (ii). For variation of c' with ω' we take $v' = 0.4$ in (ii).

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