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ANALYSIS OF SECTORIAL PLATES NORMALLY LOADED OVER CIRCULAR DOMAINS-I

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The method of images is used to obtain solutions for the small deflections of sectorial plates bounded by two simply supported radii and an elastically restrained or free circular edge, the plates being subject to three types of normal loading distributed over the area of a circular patch. Formulae for the bending and twisting moments along the circular edge are given.

1. INTRODUCTION

Problems dealing with thin plates in the form of uniformly loaded circular sectors the straight edges of which are simply supported were discussed by Nadai and Galerkin (Timoshenko and Woinowsky-Krieger 1959, p. 295). When the radial edges are clamped or free various approximate and exact methods were applied, references are to be found on page 298 of the same book. The case of a sectorial plate which is simply supported along the radial edges, free along the circular edge and normally loaded over a concentric sector was considered by the author (1960a). Solutions for singularly or uniformly loaded sectorial plates bounded by two arcs of concentric circles and two simply supported radii were also obtained by Bassali (1960 b) and Bassali and Gorgui (1960 a, b) when the circular edges are either elastically restrained or free. Methods using approximate solutions of integral equations were applied by Bassali and Halim (1961, 1963) to study the bending of uniformly loaded plates in the form of a semi-circle or a quadrant of a circle which are clamped along the radii and elastically restrained or free along the circular edges. When the same plates are clamped along the radii, free along the circular edges and acted upon by concentrated forces two methods were used by Bassali, Mahmood and Refai (1976 a, b) to establish the required solutions.

In this paper solutions for circular plates which are elastically restrained or free along their edges and are subject to diverse types of pressures over an eccentric circular patch are used to establish the appropriate solutions corresponding to sectorial plates bounded by a circular edge and two simply supported radii when the circular edge is either elastically restrained or free and the plates are normally loaded by three types of loading distributed on the surface of a circle.

2. Formulation of the Problem and Boundary Conditions

We consider a thin isotropic sectorial plate bounded by the two radii OA, OB and the circular edge Γ of radius c (see Fig. 1). Let $\hat{AOB} = \gamma = \pi/k$, where k is an integer > 1. We assume that the plate is simply supported along the radial edges

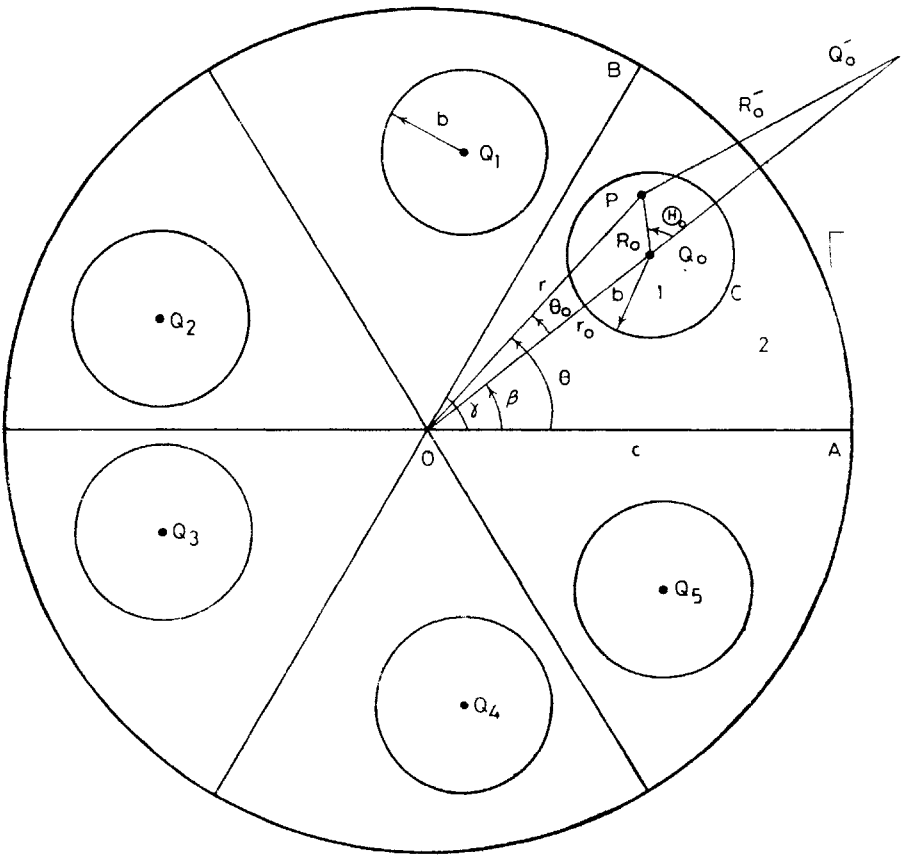

FIG. 1.

and is elastically restrained or free along Γ. Let $z = c\zeta = re^{i\theta}$ be the complex variable of any point P in the mid-plane of the plate and $Q_0(z_0 = c\zeta_0 = r_0 e^{i\beta})$ be the centre of the circle of radius $b = cq$ and boundary C, where $b \leqslant$ the smallest of $r_0 \sin\beta$, $r_0 \sin(\gamma-\beta)$ and $c - r_0$. We assume that $p_1(r, \theta)$ and $p_2 = 0$ denote the load intensities at points in regions 1 and 2 which lie inside and outside C, respectively. If w_1 and w_2 are the small deflections at points of the two regions, measured positively downwards, then

$$\nabla^4 w_1 = p_1(r, \theta)/D \qquad \ldots(1a)$$

$$\nabla^4 w_2 = 0 \qquad \ldots(1b)$$

where D is the flexural rigidity of the plate. Let (R_0, Θ_0) be the polar coordinates of P referred to Q_0 as pole and OQ_0 as initial line (see Fig. 1). We shall consider the three following cases of loading:

$$\text{Case I: } p_1 = p_0 R_0^{s-2} (s \geqslant 2) \qquad \ldots(2a)$$

$$\text{Case II: } p_1 = p_0 R_0^{s-3} \cos \Theta_0 (s \geqslant 3) \qquad \ldots(2b)$$

$$\text{Case III: } p_1 = p_0 R_0^{s-3} \sin \Theta_0 (s \geqslant 3). \qquad \ldots(2c)$$

It is required to obtain expressions for w_1 and w_2 which satisfy the following conditions: (i) w_1 satisfies (1a), (ii) w_2 satisfies (1b) and the boundary conditions along Γ and the radial edges OA, OB, (iii) w_1 and w_2 must conform to the continuity conditions across C. We shall adopt the notation

$$| \zeta | = r/c = \rho, \; | \zeta_0 | = r_0/c = \rho_0, \; d = \partial/\partial\rho, \; d' = \partial/\partial\theta. \qquad \ldots(3)$$

The two following cases of boundary conditions will be considered:

Case (*a*)—The radial edges are simply supported and Γ is elastically restrained according to

$$(w_2)_{\rho=1} = 0, \; (d^2 w_2 + \nu d w_2)_{\rho=1} = 0 \qquad \ldots(4)$$

where

$$\nu = (1 + \lambda)/(1 - \lambda) \qquad \ldots(5)$$

λ is the restraining parameter. It was shown by Bassali and Dawoud (1958) that Γ is rigidly clamped if $\nu = \infty$, $\lambda = 1$ and is simply supported if $\nu = \eta$, $\lambda = (\eta - 1)/(\eta + 1)$, η being Poisson's ratio. In any intermediate case of partial fixation we have

$$\infty > \nu > \eta, \; 1 > \lambda > (\eta - 1)/(\eta + 1). \qquad \ldots(6)$$

The boundary values of the bending and twisting moments along the elastically restrained edge Γ are furnished by

$$M_r/(\eta - \nu) = M_\theta/(1 - \eta\nu) = -(D/c^2)(d w_2)_{\rho=1} \qquad \ldots(7a)$$

$$M_{r\theta} = (1 - \eta)(D/c^2)(d' d w_2)_{\rho=1} \qquad \ldots(7b)$$

$$M_r = M_\theta/\eta = -(D/c^2)(d^2 w_2)_{\rho=1}, \; M_{r\theta} = 0 \; (\Gamma \text{ clamped}). \qquad \ldots(7c)$$

Case (*b*)—The radial edges are simply supported and Γ is free. Along the free circular edge Γ we have

$$(M_r)_{\rho=1} = 0, \; (M_\theta)_{\rho=1} = (\eta^{-1} - \eta)(D/c^2)(d^2 w_2)_{\rho=1} \qquad \ldots(8a)$$

$$(M_{r\theta})_{\rho=1} = (1 - \eta)(D/c^2) d' \{(d - 1) w_2\}_{\rho=1}. \qquad \ldots(8b)$$

3. Sectorial Plate Subject to the First Type of Normal Pressure

In this section we study the bending of the circular sector AOB bounded by the two simply supported radial edges $\theta=0$, $\theta=\gamma$ $(0\leqslant r\leqslant c)$ and the circular edge Γ which is either elastically restrained (Case a) or free (Case b), the plate being subject to the normal loading (2a) over the circular patch of centre Q_0 and radius b. It is easily seen that the deflected form of the sectorial plate is the same as that of the entire circular plate having the same boundary conditions along the circular edge and subject to normal loadings of intensities $(-1)^t p_0 R_t^s$ on the circular patches with centres $Q_t(z_t=r_0 e^{i\alpha_t})$ and equal radii b, where $R_t=PQ_t$ (Fig. 1) and

$$\left.\begin{aligned}\alpha_t &= t\gamma + \beta \quad (t = 0, 2, \ldots 2k-2)\\ \alpha_t &= (t+1)\gamma - \beta \quad (t = 1, 3, \ldots, 2k-1).\end{aligned}\right\} \qquad \ldots(9)$$

If n is a positive integer and $\theta_t = \theta - \alpha_t$ it can be proved that

$$\left.\begin{aligned}\sum_{t=0}^{2k-1}(-1)^t \cos n\theta_t &= 2k \sin n\theta \sin n\beta \text{ for } n = k, 2k, 3k, \ldots\\ &= 0 \text{ otherwise;}\end{aligned}\right\} \qquad \ldots(10a)$$

$$\left.\begin{aligned}\sum_{t=0}^{2k-1}\sin n\theta_t &= 2k \sin n\theta \cos n\beta \text{ for } n = k, 2k, 3k, \ldots\\ &= 0 \text{ otherwise.}\end{aligned}\right\} \qquad \ldots(10b)$$

We Introduce the notations

$$u_t = \zeta - \zeta_t, \; |u_t| = U_t = R_t/c; \; v_t = 1 - \bar{\zeta}_t \zeta, \; |v_t| = V_t = \rho_0 R_t'/c \qquad \ldots(11)$$

where $R_t' = PQ_t'$, $Q_t'(z_t' = c\zeta_t' = c/\bar{\zeta}_t)$ is the inverse of Q_t $(z_t = c\zeta_t)$ with respect to Γ. (See Fig. 1). The two cases of boundary conditions will now be separately considered.

Case (a)—Assuming that w_{er}^1 and w_{er}^2 denote the deflections in regions 1 and 2 of the entire circular plate subject to the loading (2a) over the circular patch with centre Q_0 and radius b it was shown by Bassali and Dawoud (1956) that*

$$w_{er}^2 = \frac{P_0 c^2}{8\pi D}\left[(U_0^2 + q'^2)\ln\frac{U_0}{V_0} + (1-\rho^2)\left\{\frac{1}{2}\lambda(1-\rho_0^2-q'^2) + (1-\rho_0^2+m'q'^2)\,\mathrm{Re}S_m(\bar{\zeta}_0\zeta)\right\}\right] \qquad \ldots(12)$$

$$[w]_2^1 = \frac{P_0}{8\pi D}\left[\left(1-\frac{1}{s}\right)\left(R_0^2-b^2\right) + \frac{4(R_0^{s'}-b^{s'})}{ss'^2 b^s} + (R_0^2+b'^2)\ln\frac{b}{R_0}\right] \qquad \ldots(13)$$

where $P_0 = 2\pi p_0 b^s/s$ is the total load on the plate and

* Notice the differenee in notation and that the deflection there is measured positively upwards.

$$s' = s + 2,\ b'^2 = sb^2/s',\ q'^2 = b'^2/c^2,\ m = (1-\lambda)^{-1},\ m' = m - 1 \qquad \ldots(14)$$

$$S_m(x) = \sum_{n=0}^{\infty} \frac{x^n}{n+m} = x^{-m}\int_0^x \frac{t^{m'}}{1-t}\,dt\ (\,|\,x\,|\,<|\,). \qquad \ldots(15)$$

For a clamped boundary Γ we have the closed form

$$w_{cl}^2 = \frac{P_0c^2}{8\pi D}\mathrm{Re}\left[(U_0^2+q'^2)\ln\frac{u_0}{v_0} + (1-\rho^2)\left\{\frac{1}{2}(1-\rho_0^2-q'^2) + \frac{q'^2}{v_0}\right\}\right]. \qquad \ldots(16)$$

Substracting (16) from (12) we find that the appropriate solutions for the sectorial plate are

$$w_{cl}'^2 = \frac{P_0c^2}{8\pi D}\sum_{t=0}^{2k-1}(-1)^t Re\left[(U_t^2+q'^2)\ln\frac{u_t}{v_t} + \frac{(1-\rho^2)q'^2}{v_t}\right] \qquad \ldots(17)$$

$$w_{cr}'^2 - w_{cl}'^2 = \frac{P_0c^2(1-\rho^2)}{8\pi D}\sum_{t=0}^{2k-1}(-1)^t\sum_{n=0}^{\infty}(\rho_0\rho)^n\{1-\rho_0^2-(n+1)q'^2\}\frac{\cos n\theta_t}{n+m} \qquad \ldots(18a)$$

$$= \frac{kP_0c^2(1-\rho^2)}{4\pi D}\sum{}'\frac{(\rho_0\rho)^n}{n+m}\left\{1-\rho_0^2-(n+1)q'^2\right\}\sin n\theta\sin n\beta \qquad \ldots(18b)$$

where equation (10a) is applied and the accent on Σ indicates that the summation is carried on over integral multiples of k. By expanding $\ln(u_t/v_t)$ and v_t^{-1} in (17) in powers of ζ, taking the real parts and using (10a) we obtain

$$w_{cl}'^2 = (kP_0c^2/4\pi D)\,\Sigma'\,f_n(\rho,\rho_0)\sin n\theta\sin n\beta \qquad \ldots(19a)$$

$$w_{cr}'^2 = (kP_0c^2/4\pi D)\,\Sigma'\,g_n(\rho,\rho_0)\sin n\theta\sin n\beta \qquad \ldots(19b)$$

$$g_n(\rho,\rho_0) = f_n(\rho,\rho_0) + (1-\rho^2)\{1-\rho_0^2-(n+1)q'^2\}(\rho_0\rho)^n/(n+m) \qquad \ldots(20a)$$

$$f_n(\rho,\rho_0) = [t_{n-1}(\rho) + \rho_0^2\rho^2 t_{n+1}(\rho) - (\rho^2+\rho_0^2)t_n(\rho) + \{1-\rho^2-t_n(\rho)\}q'^2](\rho_0\rho)^n\ (\rho>\rho_0) \qquad \ldots(20b)$$

$$f_n(\rho,\rho_0) = [t_{n-1}(\rho_0) + \rho_0^2\rho^2 t_{n+1}(\rho_0) - (\rho^2+\rho_0^2)t_n(\rho_0) + \{1-\rho^2-t_n(\rho_0)\}q'^2](\rho_0\rho)^n(\rho<\rho_0) \qquad \ldots(20c)$$

where

$$t_n(\rho) = (\rho^{-2n}-1)/n. \qquad \ldots(21)$$

Equations (12) and (13) show that at any point of region 1 the deflection w_{er}^{1} of the sectorial plate is furnished by

$$w_{er}^{1} = (P_0/8\pi D)[(R_0^2 + b'^2)\ln(b/\rho_0 R_0') + \sum_{t=1}^{2k-1}(-1)^t(R_t^2+b'^2)\ln(R_t/\rho_0 R_t')$$

$$+ (1-1/s)(R_0^2-b^2)+4(R_0^{s'}-b^{s'})/ss'^2 b^s+2kc^2(1-\rho^2)(1-\rho_0^2+m'q'^2)$$

$$\times \Sigma'(\rho_0\rho)^n \sin n\theta \sin n\beta/(n+m)]. \qquad ...(22)$$

Substitution from (19b), (20a, b) and (21) in (7a, b, c) yields

$$M_r/(\eta-\nu)=M_\theta/(1-\eta\nu)=(kP_0/2\pi)\ \Sigma'\ \rho_0^n$$

$$\times \{1-\rho_0^2-(n+1)q'^2\}\sin n\theta \sin n\beta/(n+m) \qquad ...(23a)$$

$$M_{r\theta}/(1-\eta) = (kP_0/2\pi)\ \Sigma'\ n\rho_0^n\{(n+1)q'^2 + \rho_0^2 - 1\}\cos n\theta \sin n\beta/(n+m)$$

...(23b)

$$M_r = M_\theta/\eta=(kP_0/\pi)\Sigma'\rho_0^n\{(n+1)q'^2 + \rho_0^2-1\}\sin n\theta \sin n\beta\ (\Gamma \text{ clamped}).$$

...(23c)

Case (b)—Assuming that w_{fr}^1 and w_{fr}^2 denote the deflections in regions 1 and 2 of the entire circular plate which is free along the circular edge, subject to the loading (2a) over the area of the circle with centre Q_0, radius b and supported at any number N of points it was proved by Bassali (1957, p. 735) that

$$w_{fr}^2 = (c^2/8\pi\kappa D)\left[\sum_{j=0}^{N} P_j U_j^2\ (\kappa \ln U_j - \ln V_j) + P_0 q'^2\ (\kappa \ln U_0 - \ln V_0)\right.$$

$$+ (1-\kappa^2)\ \mathrm{Re}\ I_N(\zeta) + \frac{1-\rho^2}{1+\kappa}\left\{\sum_{j=0}^{N}(1-\rho_j^2)P_j + P_0 q'^2\right.$$

$$\left.\left.\times\left(\mathrm{Re}\ \frac{1+\kappa}{\nu_0} - 1\right)\right\}\right] + \gamma_1 x + \gamma_2 y + \gamma_3 \qquad ...(24)$$

where $P_1, P_2, \ldots, P_N$ are the concentrated reactions, measured positively downwards at the N points of support $z_j = c\zeta_j$ $(j = 1,2,\ldots,N)$, $\gamma_1, \gamma_2, \gamma_3$ are real constants,

$$\kappa = (3+\eta)/(\eta-1) \qquad ...(25)$$

$$I_N(\zeta) = \sum_{j=0}^{N} P_j L_j(\zeta),\ L_j(\zeta)=\int_0^1 (1-\bar{\zeta}_j\zeta t)\ln(1-\bar{\zeta}_j\zeta t)\frac{dt}{t}. \qquad ...(26)$$

The expression for w_{fr}^1 is obtained by adding (24) to (13).

We now derive the formula for w_{fr}^2 corresponding to the sectorial plate. The vanishing of the deflection on the bounding radii leads to $\gamma_1 = \gamma_2 = \gamma_3 = 0$. Setting $P_j = 0 (j = 1,2,\ldots,N)$ and summing for all the loadings on the $2k$ eccentric circles we arrive at

$$w_{fr}^2 = \frac{P_0 c^2}{8\pi\kappa D} \sum_{t=0}^{2k-1} (-1)^t \left\{ (U_t^2 + q'^2)(\kappa \ln U_t - \ln V_t) + (1-\kappa^2) \operatorname{Re} L_t(\zeta) \right.$$

$$\left. + \operatorname{Re} \frac{(1-\rho^2)q'^2}{v_t} \right\}. \qquad \ldots(27)$$

Subtracting (17) from (27) gives

$$w_{fr}^2 - w_{cl}^2 = \frac{P_0 c^2 (\kappa - 1)}{8\pi D} \sum_{t=0}^{2k-1} (-1)^t \left\{ (U_t^2 + q'^2) \ln V_t - (1+\kappa) \operatorname{Re} L_t(\zeta) \right.$$

$$\left. - \operatorname{Re} \frac{(1-\rho^2)q'^2}{v_t} \right\}. \qquad \ldots(28)$$

Expanding $\ln v_t$, $L_t(\zeta)$, v_t^{-1} in powers of ζ, taking the real parts and applying (10a) we find

$$w_{fr}^2 = (kP_0 c^2/4\pi D) \Sigma' h_n(\rho, \rho_0) \sin n\theta \sin n\beta \qquad \ldots(29)$$

$$h_n(\rho, \rho_0) = f_n(\rho, \rho_0) + \frac{4(\rho\rho_0)^n}{3+\gamma_i} \left\{ \left(1 - \rho^2 + \frac{1}{n}\right)\left(\frac{1}{n} - \frac{\rho_0^2}{n+1} - q'^2\right) \right.$$

$$\left. - \frac{\kappa}{n^2(n-1)} \right\}. \qquad \ldots(30)$$

Substitution from (20b) and (30) in (8a,b) leads to

$$(M_\theta)_{\rho=1} = \frac{k(1+\eta)P_0}{\pi(3+\eta)} \sum{}' \left[\frac{4}{n} + (1-\eta) \left\{ 1 - \rho_0^2 - (n+1)q'^2 \right\} \right]$$

$$\rho_0^n \sin n\theta \sin n\beta \qquad \ldots(31a)$$

$$(M_{r\theta})_{\rho=1} = \frac{kP_0}{\pi\kappa} \sum{}' \left[1 - \rho_0^2 + \frac{\kappa+1}{n} - (n+1)q'^2 \right] \rho_0^2 \cos n\theta \sin n\beta. \qquad \ldots(31b)$$

Letting c tend to ∞ in any of the solutions w_{cl}^2, w_{cr}^2, w_{fr}^2 we obtain the following solution for a wedge-shaped plate simply supported along the radial edges and carrying the later load (2a) on a circular patch :

$$w'^{2} = \frac{P_0}{8\pi D} \sum_{t=0}^{2k-1} (-1)^t \left(R_t^2 + b'^2 \right) \ln \frac{R_t}{b} \qquad \ldots(32)$$

$$w'^{1} = \frac{P_0}{8\pi D} \left[\left(1 - \frac{1}{s} \right) \left(R_0^2 - b^2 \right) + \frac{4(R_0{}^{s'} - b^{s'})}{ss'^2 b^s} + \sum_{t=1}^{2k-1} \left(-1 \right)^t \left(R_t^2 + b'^2 \right) \ln \frac{R_t}{b} \right]. \qquad \ldots(33)$$

The solution (32) can be expanded in the form

$$w'^{2} = \frac{kP_0}{4\pi D} \sum{}' \frac{1}{n} \left\{ \frac{r^2}{n-1} - \frac{r_0^2}{n+1} - b'^2 \right\} \left(\frac{r_0}{r} \right)^n \sin n\theta \sin n\beta \quad (r > r_0) \qquad \ldots(34)$$

and for $r < r_0$ we interchange r and r_0 in (34).

4. Sectorial Plate Subject to the Second Type of Normal Pressure

This section deals with the sectorial plate AOB under case (a,b) of the boundary conditions but subject to the normal loading (2b) on the area of the circle with centre Q_0 and radius b.

Case(a)—The deflections w_{er}^2 and w_{er}^1 of the entire circular plate acted upon by the loading (2b) on the circular patch with centre Q_0 and radius b were obtained by the author (1956). Setting $s - 1$ instead of s, measuring the deflection positively downwards and making obvious changes in notation it was found that

$$w_{er}^2 = \frac{Mc}{16\pi\rho_0 D} \mathrm{Re} \left[2 \left(\bar{\zeta}_0\zeta + \zeta_0\bar{\zeta} - 2\rho_0^2 \right) \ln \frac{v_0}{u_0} + \left(\frac{\bar{\zeta}_0\zeta}{v_0} - \frac{\zeta_0}{u_0} \right) q'^2 - 2(1-\rho^2) \left\{ \frac{\rho_0^2}{m} + \sum_{n=1}^{\infty} \frac{(\bar{\zeta}_0\zeta)^n}{n+m} \left(m + \overline{2-m}\, \rho_0^2 - \frac{1}{2} m'nq'^2 \right) \right\} \right] \qquad \ldots(35)$$

where $M = \pi p_0 b^s/s$ is the moment of the loading (2b) about the diameter of the patch perpendicular to OQ_0. The formula for w_{er}^1 is obtained by adding (35) to the following expression which does not depend on the boundary conditions :

$$[w]_2^1 = \frac{Mc}{16\pi D} (\rho\cos\theta_0 - \rho_0) \left[4 \ln \frac{R_0}{b} + \frac{4}{s} + \frac{b'^2}{R_0^2} - \frac{R_0^2}{b_1^2} + \frac{16(R_0/b)^s}{s(s^2-4)} \right] \qquad \ldots(36)$$

where

$$b_1^2 = (s-2)\,b^2/s. \qquad \ldots(37)$$

For a clamped boundary m tends to ∞ and we have

$$w_{cl}^2 = \frac{Mc}{16\pi\rho_0 D}\,\mathrm{Re}\Big[\,4\rho_0(\rho\cos\theta_0-\rho_0)\ln\frac{v_0}{u_0} - \frac{\zeta_0 q'^2}{u_0} + \frac{\bar{\zeta}_0\zeta}{v_0}\Big\{q'^2 - 2(1-\rho_0^2)(1-\rho^2) + \frac{(1-\rho^2)\,q'^2}{v_0}\Big\}\Big]. \qquad \ldots(38)$$

Subtracting (38) from (35) and applying (10a) we obtain the following solutions for the sectorial plate:

$$w_{cl}'^2 = \frac{Mc}{16\pi\rho_0 D}\sum_{t=0}^{2k-1}(-1)^t\,\mathrm{Re}\Big[4\rho_0(\rho\cos\theta_t-\rho_0)\ln\frac{v_t}{u_t} - \frac{\zeta_t q'^2}{u_t} + \frac{\bar{\zeta}_t\zeta}{v_t}\Big\{q'^2 - 2(1-\rho_0^2)(1-\rho^2) + \frac{(1-\rho^2)q'^2}{v_t}\Big\}\Big] \qquad \ldots(39)$$

$$w_{er}'^2 - w_{cl}'^2 = \frac{kMc(1-\rho^2)}{4\pi\rho_0 D}\sum{}'\frac{(\rho_0\rho)^n}{n+m}\Big\{n(1-\rho_0^2) - 2\rho_0^2 - \frac{1}{2}n(n+1)q'^2\Big\}\sin n\theta\sin n\beta. \qquad \ldots(40)$$

Expanding all the terms in (39) in powers of ζ, taking the real parts and using (10a) we get

$$w_{cl}'^2 = (kMc/4\pi\rho_0 D)\sum{}' f_n(\rho,\rho_0)\sin n\theta\sin n\beta \qquad \ldots(41a)$$

$$w_{er}'^2 = (kMc/4\pi\rho_0 D)\sum{}' g_n(\rho,\rho_0)\sin n\theta\sin n\beta \qquad \ldots(41b)$$

$$g_n(\rho,\rho_0) = f_n(\rho,\rho_0) + (1-\rho^2)\{n(1-\rho_0^2) - 2\rho_0^2 - \tfrac{1}{2}n(n+1)q'^2\}\times(\rho_0\rho)^n/(n+m) \qquad \ldots(42a)$$

$$f_n(\rho,\rho_0) = [\,t_{n-1}(\rho) + \rho_0^2\rho^2 t_{n+1}(\rho) - 2\rho_0^2 t_n(\rho) - (1-\rho_0^2)(1-\rho^2) + \tfrac{1}{2}n\{1-\rho^2-t_n(\rho)\}q'^2\,](\rho_0\rho)^n\ (\rho>\rho_0) \qquad \ldots(42b)$$

$$f_n(\rho,\rho_0) = [\,t_{n-1}(\rho_0) + \rho_0^2\rho^2 t_{n+1}(\rho_0) - 2\rho_0^2 t_n(\rho_0) - (1-\rho_0^2)(1-\rho^2) + q'^2 + \tfrac{1}{2}n\{1-\rho^2-t_n(\rho_0)\}q'^2\,](\rho_0\rho)^n\ (\rho<\rho_0) \qquad \ldots(42c)$$

$$M_r/(\eta-\nu) = M\theta/(1-\eta\nu) = (kM/2\pi r_0)\,\Sigma'\rho_0^n\{n(1-\rho_0^2) - 2\rho_0^2 - \tfrac{1}{2}n(n+1q'^2\}\sin n\theta\sin n\beta/(n+m) \qquad \ldots(43a)$$

$$M_{r\theta}/(1-\eta) = (kM/2\pi r_0)\,\Sigma' n\rho_0^n\{n(\rho_0^2-1) + 2\rho_0^2 + \tfrac{1}{2}n(n+1)q'^2\}\cos n\theta \times \sin n\beta/(n+m) \qquad \ldots(43b)$$

$$M_r = M_\theta/\eta = (kM/\pi r_0)\,\Sigma'\rho_0^n\{n(\rho_0^2-1) + 2\rho_0^2 + \tfrac{1}{2}n(n+1)q'^2\} \times \sin n\theta\sin n\beta\ (\Gamma\text{clamped}). \qquad \ldots(43c)$$

Case(b)—It was proved by the author (1958) that the deflection w^2_{fr} of the complete circular plate which is supported at the N points $c\zeta_j$ ($j = 1,2, ..N,$) and is normally loaded by the pressure (2b) on the circular region with centre Q_0 and radius b is given by

$$w^2_{fr} = \frac{c}{8\pi\kappa D}\left[c\sum_{j=1}^{N} P_j \left\{ U_j^2 (\kappa \ln U_j - \ln V_j) + \frac{(1-\rho^2)(1-\rho_j^2)}{\kappa+1} \right.\right.$$

$$\left. + \operatorname{Re} L_j(\zeta)\right\} + M \operatorname{Re}\bar{\lambda}\left\{ 2u_0 \left(\ln V_0 - \kappa \ln U_0 - \frac{\kappa b'^2}{4R_0^2} \right)\right.$$

$$- \frac{2(1-\rho^2)\zeta_0}{k+1} + \frac{1-\kappa^2}{\zeta_0} v_0 \ln v_0 - \frac{(1-\rho^2)u_0}{v_0}$$

$$\left.\left. + \left(1 + \frac{1-\rho^2}{v_0} \right) \frac{\zeta q'^2}{2v_0} \right\}\right] + \gamma_1 x + \gamma_2 y + \gamma_3 , \qquad ...(44)$$

where

$$\lambda = e^{i\beta} = \zeta_0/\rho_0. \qquad ...(45)$$

The addition of (36) to (44) gives w^1_{fr} . The expression for w^2_{fr} is now obtained by setting $\gamma_1 = \gamma_2 = \gamma_3 = 0$, $P_1 = P_2 = ... = P_N = 0$ and summing for all the loadings on the $2k$ circular regions which keep the plate in equilibrium.

$$w'^2_{fr} = \frac{Mc}{16\pi\kappa\rho_0 D}\sum_{t=0}^{2k-1} (-1)^t \operatorname{Re}\left[4\rho_0(\rho\cos\theta_t - \rho_0)(\ln v_t - \kappa \ln u_t) - \frac{\kappa\zeta_t + q'^2}{u_t} \right.$$

$$\left. + 2(1-\kappa^2)v_t \ln v_t + \frac{\bar{\zeta}_t\zeta}{v_t}\left\{ q'^2 - 2(1-\rho_0^2)(1-\rho^2) + \frac{(1-\rho^2)q'^2}{v_t} \right\}\right]. \qquad ...(46)$$

Subtracting (39) from (46) gives

$$w'^2_{fr} - w'^2_{ct} = \frac{Mc(1-\kappa)}{16\pi\kappa\rho_0 D}\sum_{t=0}^{2k-1} (-1)^t \operatorname{Re}\left[4\rho_0 (\rho\cos\theta_t - \rho_0)\ln v_t \right.$$

$$+ 2(1+\kappa)v_t \ln v_t + \frac{q'^2 - 2(1-\rho_0^2)(1-\rho^2)}{v_t}$$

$$\left. + \frac{(1-\rho^2)\bar{\zeta}_t\zeta q'^2}{v_t^2} \right] . \qquad ...(47)$$

Thus we have

$$w'^2_{fr} = (kMc/4\pi\rho_0 D)\,\Sigma'\, h_n(\rho,\rho_0)\sin n\theta \sin n\beta \qquad ...(48a)$$

$$h_n(\rho,\rho_0) = f_n(\rho,\rho_0) + \frac{4(\rho_0\rho)^n}{3+\eta}\left[\left(1-\rho^2+\frac{1}{n}\right)\left(1-\frac{n+2}{n+1}\rho_0^2-\frac{1}{2}nq'^2\right)\right.$$
$$\left.-\frac{\kappa}{n(n-1)}\right] \quad (0\leqslant\rho\leqslant 1) \qquad \ldots(48b)$$

$$(M_\theta)_{\rho=1} = \frac{kM(1+\eta)}{\pi(3+\eta)r_0}\sum{}'\left[4+(1-\eta)\left\{n-(n+2)\rho_0^2\right.\right.$$
$$\left.\left.-\frac{1}{2}n(n+1)q'^2\right\}\right]\rho_0^n \sin n\theta \sin n\beta \qquad \ldots(49a)$$

$$(M_{r\theta})_{\rho=1} = \frac{kM}{\pi\kappa r_0}\sum{}'\left[n+\kappa+1-(n-2)\rho_0^2-\frac{1}{2}n(n+1)q'^2\right]\rho_0^n$$
$$\cos n\theta \sin n\beta. \qquad \ldots(49b)$$

Letting c tend to ∞ in (39) yields the following solution for wedge-shaped plate :

$$w'^2 = (M/4\pi D)\sum_{t=0}^{2k-1}(-1)^t(r_0-r\cos\theta_t)\{\ln(R_t/b)+(b'^2/4)R_t^2\} \qquad \ldots(50a)$$

$$w'^1 = (M/4\pi D)\left[(r\cos\theta_0-r_0)\{s^{-1}-R_0^2/4b_1^2+4(R_0/b)^s/s(s^2-4)\}\right.$$
$$\left.+\sum_{t=1}^{2k-1}(-1)^t(r_0-r\cos\theta_t)\{\ln(R_t/b)+b'^2/4R_t^2\}\right] \qquad \ldots(50b)$$

$$w'^2 = \frac{kM}{4\pi r_0 D}\sum{}'\left[\frac{r^2}{n-1}-\frac{n+2}{n(n+1)}r_0^2-\frac{1}{2}b'^2\right]\left(\frac{r_0}{r}\right)^n$$
$$\sin n\theta \sin n\beta \ (r>r_0) \qquad \ldots(51a)$$

$$w'^2 = \frac{kM}{4\pi r_0 D}\sum{}'\left[\frac{r^2}{n+1}-\frac{n-2}{n(n-1)}r_0^2+\frac{1}{2}b'^2\right]\left(\frac{r}{r_0}\right)^n$$
$$\sin n\theta \sin n\beta \ (r<r_0). \qquad \ldots(51b)$$

Setting $q' = 0$ in the results of this section gives the solutions corresponding to a couple nucleus M acting at Q_0 in a plane normal to the plate through OQ_0.

5. Sectorial Plate Subject to the Third Type of Normal Pressure

Solutions will now be derived for the sectorial plate under the normal loading (2c) on the cirular domain.

Case (a)— Following a procedure similar to that adopted by the author (1956) we easily obtain the following solution for the elastically restrained circular plate under the load (2c) on the circular domain with centre Q_0 and radius b:

$$w'^2_{er} = \frac{Mc}{8\pi\rho_0 D}\operatorname{Im}\left[(\bar\zeta_0\zeta-\zeta_0\bar\zeta)\ln\frac{v_0}{u_0}+\frac{1}{2}\left(\frac{\zeta}{u_0}+\frac{1}{v_0}\right)q'^2\right.$$
$$\left.+(1-\rho^2)\left\{\frac{m'q'^2}{v_0}-m\left(1-\rho_0^2\frac{1}{2}m'q'^2\right)S_m(\bar\zeta_0\zeta)\right\}\right] \qquad \ldots(52)$$

$$[w]_2^1 = \frac{Mr\sin\theta}{4\pi D}\left[\ln\frac{R_0}{b} + \frac{1}{s} + \frac{b'^2}{4R_0^2} - \frac{R_0^2}{4b_1^2} + \frac{4(R_0/b)^s}{s(s^2-4)}\right] \quad ...(53)$$

where $S_m(x)$ is defined by (15). For a clamped edge we have

$$w_{cl}^2 = \frac{Mc}{8\pi\rho_9 D}\,\mathrm{Im}\left[(\bar{\zeta}_0\zeta - \zeta_0\bar{\zeta})\ln\frac{v_0}{u_0} + \frac{1}{2}\left(\frac{\zeta}{u_0} + \frac{1}{v_0}\right)q'^2 + \frac{1-\rho^2}{v_0}\left(\rho_0^2 - 1 + \frac{\bar{\zeta}_0\zeta q'^2}{2v_0}\right)\right]. \quad ...(54)$$

Subtracting (54) from (52) we get

$$w_{er}^2 - w_{cl}^2 = \frac{Mc(1-\rho^2)}{8\pi\rho_0 D}\sum_{n=1}^{\infty}\frac{n(\rho_0\rho)^n}{n+m}\left\{1-\rho_0^2 - \frac{1}{2}(n+1)q'^2\right\}\sin n\theta. \quad ...(55)$$

It can be seen that the deflected form of the sectorial plate AOB under the pressure (2c) is the same as that of the complete circular plate acted upon by normal pressures of intensities $p_0 R_t^{s-3}\sin\Theta_t$ on the $2k$ eccentric circles.

We therefore have

$$w_{cl}'^2 = \frac{Mc}{8\pi\rho_0 D}\sum_{t=0}^{3k-1}\mathrm{Im}\left[(\bar{\zeta}_t\zeta - \zeta_t\bar{\zeta})\ln\frac{v_t}{u_t} + \frac{1}{2}\left(\frac{\zeta}{u_t} + \frac{1}{v_t}\right)q'^2 + \frac{1-\rho^2}{vt}\left(\rho_0^2 - 1 + \frac{\bar{\zeta}_t\zeta q'^2}{2v_t}\right)\right]. \quad ...(56)$$

Expanding in powers of ζ, taking the imaginary parts and applying (10a) we find

$$w_{cl}'^2 = (kMc/4\pi\rho_0 D)\,\Sigma'\, f_n(\rho,\rho_0)\sin n\theta\cos n\beta \quad ...(57a)$$

$$w_{er}'^2 = (kMc/4\pi\rho_0 D)\,\Sigma'\, g_n(\rho,\rho_0)\sin n\theta\cos n\beta \quad ...(57b)$$

$$g_n(\rho,\rho_0) = f_n(\rho,\rho_0) + n(1-\rho^2)\{1-\rho_0^2-\tfrac{1}{2}(n+1)q'^2\}(\rho_0\rho)^n/(n+m) \quad ...(58a)$$

$$f_n(\rho,\rho_0) = [\,t_{n-1}(\rho) - \rho_0^2\rho^2\,t_{n+1}(\rho) - (1-\rho_0^2)(1-\rho^2) + \tfrac{1}{2}n\{1-\rho^2-t_n(\rho)\}q'^2\,](\rho_0\rho)^n\ (\rho>\rho_0) \quad ...(58b)$$

$$f_n(\rho,\rho_0) = [t_{n-1}(\rho_0) - \rho_0^2\rho^2\,t_{n+1}(\rho_0) - (1-\rho_0^2)(1-\rho^2) + \tfrac{1}{2}n\{1-\rho^2-t_n(\rho_0)\}q'^2\,](\rho_0\rho)\ (\rho<\rho_0) \quad ...(58c)$$

$$M_r/(\eta-\nu) = M_\theta/(1-\eta\nu) = (kM/2\pi r_0)\,\Sigma'\, n\{1-\rho_0^2-\tfrac{1}{2}(n+1)q'^2\}\rho_0^n \sin n\theta\cos n\beta/(n+m) \quad ...(59a)$$

$$M_{r\theta}/(1-\eta) = (kM/2\pi r_0)\,\Sigma' n^2 \{\rho_0^2 - 1 + \tfrac{1}{2}(n+1)q'^2\}\rho_0^n \cos n\theta \cos n\beta \quad (n+m) \qquad \ldots(59b)$$

$$M_r = M_\theta^\lambda/\eta = (kM/\pi r_0)\,\Sigma' n \{\rho_0^2 - 1 + \tfrac{1}{2}(n+1)q'^2\}\rho_0^n \sin n\theta \cos n\beta \quad (\Gamma \text{ clamped}). \qquad \ldots(59c)$$

Case (b)—In this case the soution w_{fr}^2 for the entire circular plate which is supported at the N points $c\zeta_j$ ($j=1,2,\ldots,N$) and is acted upon by the load (2c) on the circular domain with centre Q_0 and radius b is obtained by setting

$$\lambda = e^{i(\beta+\pi/2)} = ie^{i\beta} = i\,\zeta_0/\rho_0$$

in (44). Putting $\gamma_1 = \gamma_2 = \gamma_3 = 0$, $P_1 = P_2 = \ldots = P_N = 0$ and summing for the loadings on the $2k$ circular domains we get

$$w_{fr}^{'2} = \frac{Mc}{8\pi\kappa\rho_0 D} \sum_{t=0}^{2k-1} \mathrm{Im}\left[(\bar{\zeta}_t\zeta - \zeta_t\bar{\zeta})(\ln v_t - \kappa \ln u_t) + \frac{\kappa\zeta q'^2}{2u_t} + (1-\kappa^2)\, v_t \ln v_t + \frac{q'^2 - 2(1-\rho_0^2)(1-\rho^2)}{2v_t} + \frac{(1-\rho^2)\,\bar{\zeta}_t\zeta q'^2}{2v_t^2} \right]. \qquad \ldots(60)$$

Subtracting (56) from (60) yields

$$w_{fr}^{'2}\; w_{ci}^{'2} = \frac{Mc(1-\kappa)}{8\pi\kappa\rho_0 D} \sum_{t=0}^{2k-1} \mathrm{Im}\left[(\bar{\zeta}_t\zeta - \zeta_t\bar{\zeta}) \ln v_t + (1+\kappa)\, v_t \ln v_t + \frac{q'^2 - 2(1-\rho_0^2)(1-\rho^2)}{2v_t} + \frac{(1-\rho^2)\,\bar{\zeta}_t\zeta q'^2}{2v_t^2} \right]. \qquad \ldots(61)$$

When this expression is expanded in powers of ζ, the imaginary parts are taken and equation (10b) is used the following fromulae are obtained :

$$w_{fr}^{'2} = \frac{kMc}{4\pi\kappa\rho_0 D} \sum{}' h_n(\rho,\rho_0) \sin n\theta \cos nd\beta \qquad \ldots(62a)$$

$$h_n(\rho,\rho_0) = f_n(\rho\rho_0) + \frac{4(\rho,\rho_0)^n}{3+\eta}\left[\left(1-\rho^2+\frac{1}{n}\right)\left(1-\frac{n\rho_0^2}{n+1}-\frac{1}{2}nq'^2\right) - \frac{\kappa}{n(n-1)}\right] \qquad \ldots(62b)$$

$$(M_\theta)_{\rho=1} = \frac{kM(1+\eta)}{\pi(3-\eta)r_0} \sum{}' \left[4 + (\eta-1)\left\{\rho_0^2\, 1 + \frac{1}{2}(n+1)q'^2\right\}\right]\rho_0^n \sin n\theta \cos n\beta \qquad \ldots(63a)$$

$$(M_{r\theta})_{\rho=1} = \frac{kM}{\pi r_0} \sum{}' \left[n + \kappa + 1 - n\ \rho_0^2 - \frac{1}{2}\, n(n+1)q'^2 \right] \rho_0^n$$

$$\cos n\theta \cos n\beta. \qquad \ldots(63b)$$

For a wedge-shaped plate we have

$$w'^2 = -\frac{Mr}{4\pi D} \sum_{t=0}^{2k-1} \left(\ln \frac{R_t}{b} + \frac{b'^2}{4R_t^2} \right) \sin \theta_t \qquad \ldots(64a)$$

$$w'^1 = \frac{Mr}{4\pi D} \left[\left\{ \frac{1}{s} - \frac{R_0^2}{4b_1^2} + \frac{4(R_0/b)^s}{s(s^2-4)} \right\} \sin \theta_0 \right.$$

$$\left. - \sum_{t=1}^{2-k_1} \left(\ln \frac{R_t}{b} + \frac{b'^2}{4R_t^2} \right) \sin \theta_t \right. \qquad \ldots(64b)$$

$$w'^2 = \frac{kM}{4\pi r_0 D} \sum{}' \left(\frac{r^2}{n-1} - \frac{r_0^2}{n+1} - \frac{1}{2} b'^2 \right) \left(\frac{r_0}{r} \right)^n$$

$$\sin n\theta \cos n\beta \ (r > r_0) \qquad \ldots(65a)$$

$$w'^2 = \frac{kM}{4\pi r_0 D} \sum{}' \left(\frac{r_0^2}{n-1} - \frac{r^2}{n+1} - \frac{1}{2} b'^2 \right) \left(\frac{r}{r_0} \right)^n$$

$$\sin n\theta \cos n\beta \ (r < r_0). \qquad \ldots(65b)$$

Setting $q' = 0$ in the results of this section gives the appropriate solutions when the sectorial plate is acted upon normally at Q_0 by a concentrated couple M with axis along OQ_0.

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