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ON THE THEORY OF APPROXIMATION OF CONTINUOUS FUNCTIONS OF INTERPOLATORY POLYNOMIALS

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§1. INTRODUCTION

Kis and Vertesi (1967) constructed the polynomials $P_n(f, x)$ of degree at most $4n$ which interpolate the given function $f(x) \in C[-1, 1]$ at the points

$$x_{kn} = \cos \frac{2k\pi}{2n+1}, \quad k = \overline{0, n}^* \qquad ...(1.1)$$

and satisfy Timan's inequality

$$| f(x) - P_n(f, x) | \leqslant C_1 \omega_f(\Delta_n(x)), \quad -1 < x < 1. \qquad ...(1.2)$$

Here $\omega_f(\,.\,)$ is the modulus of continuity of $f(x)$.

$\Delta_n(x) = n^{-1}\sqrt{1-x^2} + |x| n^{-2}$ and C_1 an absolute positive constant. We observe that still by interpolating at one end of the interval the estimate is not improved at this point considerably. Therefore, if we delete this point even then we get the same result.

Our aim in this paper is to prove Timan's (1960) well known theorem by constructing the polynomials $R_n(f, x)$ which interpolate the function at the points

$$x_{kn} = \cos \frac{2k-1}{2n} \pi, \; k = \overline{1, n}. \qquad ...(1.3)$$

We shall see that our polynomials are simple in nature than the polynomials of Kis and Vertesi (1967).

§2. CONSTRUCTION OF THE POLYNOMIALS $R_n(f, x)$

Let $-1 \leqslant x \leqslant 1$, $\cos t = x$, $\cos t_{kn} = x_{kn}$, with

$$t_k = \frac{2k-1}{2n} \pi, \; k = \overline{1, n}. \qquad ...(2.1)$$

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*$K = \overline{0, n}$ stands for $K = 0, 1, 2 \ldots n$.

Further for $k = \overline{1, 2n}$, let

$$l_k(t) = \frac{\sin n(t-t_k)\cos\left(\frac{t-t_k}{2}\right)}{2n \sin \frac{t-t_k}{2}}$$

$$= \frac{1}{2n}\left[1 + 2\sum_{j=1}^{n-1}\cos j(t-t_k) + \cos n(t-t_k)\right] \qquad \ldots(2.2)$$

and

$$s_k(t) = 4\, l_k^2(t) - 3\, l_k^4(t). \qquad \ldots(2.3)$$

Then for any function $f(x)$ given on $[-1, 1]$ we define the polynomials

$$R_n(f, x) = \frac{1+x}{2} f(1) + \frac{1-x}{2} f(-1)$$

$$+ \sum_{k=1}^{n}\left[f(x_k) - \left\{\frac{1+x}{2} f(1) + \frac{1-x}{2} f(-1)\right\}\right] r_k(x) \qquad \ldots(2.4)$$

where

$$r_k(x) = s_k + s_{2n+1-k}, \quad k = \overline{1, n}. \qquad \ldots(2.5)$$

We note that our polynomials $R_n(f, x)$ are of degree at most $4n+1$ as the fundamental polynomials $r_k(x)$ are of degree $4n$ at most, which can easily be seen from their definitions. Moreover, they interpolate the function at the points (1.3). With the help of the polynomials $R_n(f, x)$ we shall prove the following

Theorem 1— Let $f(x) \in C[-1, 1]$ and n be a natural number. Then for every $x \in [-1, 1]$, we have

$$|R_n(f, x) - f(x)| \leqslant C_2\, \omega_f(\Delta_n(x)).$$

To prove the theorem we need the following:

§3. LEMMA 1— $$\sum_{k=1}^{2n} s_k(t) = 1 - \frac{1}{64 n^3}(9 + 12\cos 2nt + 3\cos 4nt)$$

$$= 1 - \frac{3}{8n^3}\cos^4 nt \text{ for } 0 \leqslant t \leqslant \pi. \qquad \ldots(3.1)$$

PROOF: Following Kis and Vertesi (1967), we have from (2.2)

$$2n\, l_0(t) = 1 + 2\sum_{j=1}^{n-1}\cos jt - \cos nt = \sum_{j=-n}^{n} Z^j - \cos nt \qquad \ldots(3.2)$$

$$(2n)^3 \sum_{k=1}^{2n} l_k^3(t) = \sum_{k=1}^{2n} \{ \sum_{j=-3n}^{3n} C_{j,3} Z^j e^{-ijt_k} - 3\cos n(t-t_k) \sum_{j=-2n}^{2n} C_{j,2} Z^j e^{-ijt_k} + 3\cos^2 n(t-t_k) \sum_{j=-n}^{n} Z^j e^{-ijt_k} \cos^3 n(t-t_k) \}. \quad \ldots(3.3)$$

Now if we use the technique of Srivastava (1979) and observe that we now have $t_k - \pi/2n$ instead of t_k, then we easily get the sum provided we put $t - \pi/2n$, in place of t. Thus we have

$$(2n)^2 \sum_{k=1}^{2n} l_k^3(t) = C_{0,3} + 2C_{2n,3}\cos 2n\left(t - \frac{\pi}{2n}\right) - 3.2\, C'_{n,3}\cos^2 n\left(t - \frac{\pi}{2n}\right) + 3\cos^2 n\left(t - \frac{\pi}{2n}\right) \quad \ldots(3.4)$$

where $C_{0,3}$ and $C_{2n,3}$ are respectively coefficients of Z^0 and Z^{2n} for $m = 3$, $C'_{n,3}$ the coefficient of Z^n for $m = 2$ in the expansion

$$\sum_{j=-mn}^{mn} C_{j,m} Z^j = Z^{-mn}(1-Z^{2n+1})^m \sum_{j=0}^{\infty} \frac{(j+1)(j+2)\ldots(j+m-1)}{(m-1)!}. \quad \ldots(3.5)$$

Obviously $C_{j,m} = C_{-j,m}$, $j = \overline{1, mn}$.

Thus we have, after simplification

$$C_{0,3} = 3n^2 + 3n + 1,\ C_{2n,3} = \frac{1}{2}(n^2 + 3n + 2),\ C'_{n,3} = n + 1.$$

On substituting these values of $C_{0,3}$, $C_{2n,3}$ and $C'_{n,3}$ in (3.4) we obtain

$$(2n)^2 \sum_{k=1}^{2n} l_k^3(t) = \left(3n^2 - \frac{1}{2}\right) + \left(n^2 + \frac{1}{2}\right)\cos 2n\left(t - \frac{\pi}{2n}\right)$$

$$= (3n^2 - \tfrac{1}{2}) - (n^2 + \tfrac{1}{2})\cos 2nt. \quad \ldots(3.6)$$

In the same way, we have

$$(2n)^3 \sum_{k=1}^{2n} l_k^4(t) = C_{0,4} + 2C_{2n,4}\cos 2n\left(t - \frac{\pi}{2n}\right) + 2C_{4n,4}\cos 4n$$

(*equation continued on p.* 1044)

$$\times \left(t - \frac{\pi}{2n}\right) - 4 \cos n\left(t - \frac{\pi}{2n}\right) \left[2C'_{n,4} \cos n \right.$$

$$\times \left(t - \frac{\pi}{2n}\right) + \left. 2C'_{3n,4} \cos 3n \left(t - \frac{\pi}{2n}\right) \right] + 6 \cos^2 n$$

$$\times \left(t - \frac{\pi}{2n}\right) \left[C''_{0,4} + 2C''_{2n,4} \cos 2n \left(t - \frac{\pi}{2n}\right) \right.$$

$$\left. + 8 \cos^4 n \left(t - \frac{\pi}{2n}\right) + \cos^4 n \left(t - \frac{\pi}{2n}\right) \right] \quad \ldots(3.7)$$

where, again, the numbers $C_{0,4}$. $C_{2n,4}$ and $C_{4n,4}$ are respectively the coefficients of Z^0, Z^{2n} and Z^{4n} for $m = 4$ the numbers $C'_{n,4}$ and $C'_{3n,4}$ the coefficients of Z^n and Z^{3n} for $m = 3$ and the numbers $C''_{0,4}$ and $C''_{2n,4}$ the coefficients of Z^0 and Z^{2n} for $m = 2$ in the expansion (3.5).

Thus we have on simplification

$$C_{0,4} = \tfrac{1}{3}(16n^3 + 24n^2 + 3),\ C_{2n,4} = \tfrac{1}{3}(4n^3 + 12n^2 + 11n + 3)$$

$$C_{4n,4} = 1,\ C'_{n,4} = 2n^2 + 3n + 1,\ C'_{3n,4} = 1,\ C''_{0,4} = 2n + 1\ \ C''_{2n,4} = 1.$$

Hence, with these values of the coefficients, we obtain from (3.7)

$$(2n)^3 \sum_{k=1}^{2n} l_k^4(t) = \left(\frac{16}{3}n^3 - \frac{4}{3}n + \frac{3}{8}\right) - \left(\frac{8n^3}{3} + \frac{4n}{3} - \frac{1}{2}\right)\cos 2nt$$

$$+ \tfrac{1}{8} \cos 4nt. \quad \ldots(3.8)$$

From (3.8) and (3.6) we have

$$\sum_{k=1}^{2n} [4 l_k^3(t) - 3 l_k^4(t)] = 1 - \frac{1}{64n^3}(9 + 12 \cos 2nt + 3 \cos 4nt) \quad \ldots(3.9)$$

which is essentially the same as (3.1).

Remark : We can compare this identity with the identity of Tureckii (1965) where we have

$$\sum_{k=1}^{2n} 4 l_k^3(t) - 3 l_k^4(t) \equiv 1,$$

when the nodes of interpolation are (1.1) σ

Lemma 2— $| l_k^3(t) + l_{k+1}^3(t) | \leqslant \frac{3\pi}{\sigma_k^4}$

where $\sigma_k = 2n \sin\left(\frac{t-t_k}{2}\right)$.

PROOF: Since

$$l_k^3 + l_{k+1}^3 = (l_k + l_{k+1})(l_k^2 - l_k l_{k+1} + l_{k+1}^2)$$

and

$$l_k + l_{k+1} = \frac{(-1)^{k-1} \cos nt \sin \frac{\pi}{2n}}{2n \sin \frac{t-t_k}{2} \sin \frac{t-t_{k+1}}{2}} \quad \ldots(3.10)$$

Hence, we have

$$|l_k^3(t) + l_{k+1}^3(t)| \leqslant \frac{\pi}{|\sigma_k| |\sigma_{k+1}|}\left[\sigma_k^{-2} + \frac{1}{|\sigma_k| |\sigma_{k+1}|} + \sigma_{k+1}^{-2}\right]$$

$$\leqslant \frac{3\pi}{\sigma_k^4}$$

since $|\sigma_k| \leqslant |\sigma_{k-1}|$, and this proves our lemma.

§ 4. PROOF OF THE THEOREM: On accout of (2.4) we have the identity

$$R_n(f, x) - f(x) = \frac{1+x}{2}(f(1) - f(x))$$

$$+ \frac{1-x}{2}(f(-1) - f(x))\left[1 - \sum_{k=1}^{n} r_k(x)\right]$$

$$+ \sum_{k=1}^{n} [f(x_k) - f(x)]\, r_k(x)$$

$$= S_1 + S_2 \text{ (say)}. \quad \ldots(4.1)$$

Using the properties of modulus of continuity we have,

$$(1+x)\,\omega_f(1-x) + (1-x)\,\omega_f(1+x) \leqslant \sigma\,\omega_f(1-x^2),\ x \in [-1, 1]. \quad \ldots(4.2)$$

Hence, we obtain after using (4.2) and (3.1)

$$|S_1| \leqslant \frac{3}{8n^3} \cdot \frac{\sigma}{2}\,\omega_f(1-x^2)$$

$$\leqslant \frac{9}{8}\omega_f(\Delta_n(x)). \quad \ldots(4.3)$$

Now we break the sum S_2 into four parts after making use of (2.5), i.e.

$$S_2 = [f(\cos t_j) - f(\cos t)]\, s_j(t)$$

$$+ \sum_{k=1}^{j-1} [f(\cos t_k) - f(\cos t)]\, 4\, l_k^3(t)$$

$$+ \sum_{k=j+1}^{2n} [f(\cos t_k) - f(\cos t)]\, 4\, l_k^3(t)$$

$$- 3 \sum_{\substack{k=1 \\ k \neq j}}^{2n} [(f(\cos t_k) - f(\cos t)]\, l_k^4(t)$$

$$= [f(\cos t_j) - f(\cos t)]\, s_j(t) + S_{21} + S_{22} + S_{23} \text{ (say)}$$

where j is defined by

$$| t - t_j | \leqslant \frac{\pi}{2n}. \qquad \ldots(4.4)$$

We will now show that each constituent of the sum S_2 has the order $O\{\omega_f(\Delta_n(x))\}$.

For the first constituent, we have from

$$\cos t_k - \cos t = 2 \sin t. \sin\left(\frac{t-t_k}{2}\right)\cos\left(\frac{t-t_k}{2}\right) - 2 \cos t. \sin^2\frac{t-t_k}{2} \qquad \ldots(4.5)$$

and from (4.4) and (2.3) and the properties of modulus of continuity

$$| f(\cos t_j) - f(\cos t) | \quad | s_j(t) |$$

$$\leqslant \left(1 + \frac{\pi}{2}\right)\omega_f\left(\frac{\sqrt{1-x^2}}{n}\right) + \left(1 + \frac{\pi^2}{8}\right)\omega_f\left(\frac{|x|}{n^2}\right). \qquad \ldots(4.6)$$

The estimates for S_{21} and S_{22} are the same, so we do only S_{21}. We make use of the method of Kis and Vertesi (1967) i.e. group the summands into pairs. Thus if the number of terms in S_{21} is even, then they are grouped in pairs and no term is left but if the number of terms is odd, one term will be left out which can be estimated as (4.6). Now using Lemma 2, (4.5) and

$$\cos t_k - \cos t_{k+1} = 2 \sin \frac{\pi}{2n} \sin\left(\frac{t_k + t_{k+1}}{2}\right)$$

$$\leqslant \frac{\pi}{n}\left[\sin t. \cos(t_{k+1}-t) + 2 \cos t \sin(t_{k+1}-t)/2 \cos\left(\frac{t_{k+1}-t}{2}\right)\right] \qquad \ldots(4.7)$$

we obtain

$$| S_{21} | \leqslant 4.3\pi \sum_{k=1}^{j=1} \left(\frac{1}{\sigma_k^4} + \frac{1}{| \sigma_k |^3} \right) \omega_f \left(\frac{\sqrt{1-x^2}}{2} \right)$$

$$+ 4.3\pi \, \omega_f \left(\frac{| x |}{n^2} \right) \sum_{k=1}^{j=1} \left[\frac{1}{\sigma_{k+1}^3} + \frac{1}{2\sigma_{k+1}^2} \right]$$

since

$$| \sigma_k | = | 2n \sin \frac{t-t_k}{2} |$$

$$> \frac{2n\,[2 \mid k-j \mid -1]}{2n}$$

$$= 2i-1, \quad i = | k-j |, \; k \neq j$$

we easily get

$$| S_{21} | \leqslant 2\pi \, \omega_f \left(\frac{\sqrt{1-x^2}}{n} \right) \sum_{i=1}^{\infty} [(2i-1)^{-3} + (2i-1)^{-4}]$$

$$+ 12\pi \, \omega_f \left(\frac{| x |}{n^2} \right) \sum_{i=1}^{\infty} \left[(2i+1)^{-3} + \frac{(2i+1)^{-2}}{2} \right]$$

$$= O [\omega t \, (\Delta_n (x))]. \qquad \ldots(4.8)$$

For the last constituent, using (4.5), (4.7) and the estimates sor $l_k'(t)$, we have

$$| S_{23} | \leqslant 3 \sum_{k=1}^{2n} [| \sigma_k |^{-3} + \sigma_k^{-4}] \, \omega_f \left(\frac{\sqrt{1-x^2}}{n} \right)$$

$$+ \sum_{\substack{k=1 \\ k \neq j}}^{2n} [s_k^{-2} + 2s_k^{-2}] \, \omega_f \left(\frac{| x |}{n^2} \right)$$

$$= O \left[\omega_f \left(\frac{\sqrt{1-x^2}}{n} + \frac{| x |}{n^2} \right) \right]. \qquad \ldots(4.9)$$

This combining (4.9), (4.8) and (4.6) we obtain from (4.1)

$$| R_n (f, x) - f(x) | = O \left[\omega_f \left(\frac{\sqrt{1-x^2}}{n} + \frac{| x |}{n^2} \right) \right]$$

which proves our theorem.

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