Indian J. pure appl. Math., **14**(8) : 1056–1074, August 1983

INFLUENCE OF COUPLE STRESSES ON SINGULAR STRESS CONCENTRATION IN MICROELASTIC SOLIDS

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(*Received* 28 *August* 1982)

Based on the theory of micropolar elasticity, closed form solutions are presented for the singular stresses in the problems of half plane (i) under distributed normal and tangential loads, (ii) under concentrated normal and tangential loads, and (iii) discontinuous normal and tangential surface tractions. We observe that the convential stresses, which are unbounded in the corresponding classical solution, remain singular. The couple stresses are found to be either finite throughout the body or to possess singularities that are weaker than those encountered in classical elasticity theory. The nonvanishing of microrotation characterising ratio (b/c) is found to have a strong influence on the stresses at points sufficiently close to the source of stress concentration. The strength of singularity depends strongly on microrotation. The comparison of these results with those of Mindlin's couple stress theory and those of the classical elasticity is exhibited graphically.

1. INTRODUCTION

Muki and Sternberg (1965) investigated the problems of half plane subjected to distributed loads, concentrated normal and tangential loads, and discontinuous normal and tangential surface tractions. These problems were examined mainly in the context of couple stress theory formulated earlier by Mindlin and Tiersten (1962). Cowin (1969) studied the same problems from Cosserat's point of view as developed by Schaefer (1962). In this paper, we discuss these problems within the scope of Eringen's (1966) micropolar model of elasticity. It is worthwhile to study the influence of couple stresses in problems, which according to classical elasticity theory give rise to infinite stress concentrations with a view to ascertain the extent to which such pathologies are altered, mitigated, or possibly eliminated by the use of microelasticity. The theory of microelasticity does not leave antisymmetric part of the stress tensor indeterminate. Instead, it defines microrotation kinematically independent of the linear displacement.

The basic equations governing the micropolar theory of elasticity are recapitulated in section 2. In section 3, we have solved the problem of half plane under the distributed normal and tangential load. From this solution, we work out in section 4 the problem of half plane when the normal and tangential load is a concentrated load. Section 5 is devoted to the consideration of concentration due to discontinuous tangential loads. The numerical values of the singular stress t_{22} based on the micropolar model are than evaluated, tabulated, graphed and compared with the existing numerical results of Muki and Sternberg (1965) based on Mindlin and Tiersten model (1962).

We find that microrotation exercises a strong influence upon the stresses at points sufficiently close to the the source of stress concentration. The influence of couple stresses on the values of the singular stress t_{22} at the boundary, especially in the vicinity of the the singular point $x_2 = a$, is significant although the order of the singularity inherent in t_{22} is somehow not altered.

2. Basic Equations of Microelasticity

Equation of Motion

$$t_{kl,k} + \rho F_l = \rho \ddot{u}_l. \quad \ldots(2.1)$$

Constitutive equations

$$t_{kl} = \lambda u_{r,r}\, \delta_{kl} + \mu\,(u_{k,l} + u_{l,k}) + \chi\,(u_{l,k} - \epsilon_{klr}\Phi_r) \quad \ldots(2.2)$$

$$m_{kl} = \alpha\Phi_{r,r}\, \delta_{kl} + \beta\Phi_{k,l} + \gamma\Phi_{l,k}. \quad \ldots(2.3)$$

Equation of first stress moments

$$m_{kr,r} + \epsilon_{klr}\, t_{lr} + \rho L_k = I\dot{\sigma}_k \quad \ldots(2.4)$$

$$\dot{\sigma}_k = J\ddot{\Phi}_k. \quad \ldots(2.5)$$

In addition, the following inequalities must hold

$$3\lambda + 2\mu + \chi \geqslant 0,\ \mu \geqslant 0,\ \chi \geqslant 0,\ 3\alpha+2\gamma \geqslant 0,\ -\gamma \leqslant \beta \leqslant \gamma,\ \gamma \geqslant 0. \quad (2.6)$$

In the above equations, t_{kl} denotes the stress tensor, m_{kl} the couple stress tensor, Φ_r the microrotation vector, u_i the displacement vector, F_i the body force per unit volume, L_k the first body moment per unit mass, ρ the mass density, and σ_i the microrotation inertia of the elements. We have used the dot to denote differentiation with respect to time. Here, λ and μ are elastic constants, and χ, α, β, γ, are four additional constants due to the microstructure of the medium. The quantity J is another elastic constant.

3. Half Plane under Distributed Normal and Tangential Load

We consider the plane strain problem of half plane subjected to arbitrarily distributed ordinary surface tractions, the couple tractions being assumed to vanish over the entire bounding edge. Let D, be the open half-plane $(0 < x_1 < \infty, -\infty < x_2 < \infty)$ so that the edge L_1 is the straight line $x_1 = 0$. Depending on whether ths given loads are normal (Case A) or tangential (Case B) to the bounding edge, the boundary conditions are :

Case A :

$$t_{11}(0, x_2) = -p(x_2);\ t_{12}(0, x_2) = \sigma_1(0, x_2) = 0$$

$$\text{for } -\infty < x_2 < \infty. \qquad \text{..(3.1)}$$

Case B :

$$t_{12}(0, x_2) = p(x_2);\ t_{11}(0, x_2) = \sigma_1(0, x_2) = 0$$

$$\text{for } -\infty < x_2 < \infty. \qquad \text{...(3.2)}$$

In either case, we assume the given load p to be sectionally smooth and absolutely integrable on $(-\infty, \infty)$. Since D_1 is a bounded region, condititons (3.1) and (3.2) must be supplemented by the regularity conditions at infinity :

$$t_{\alpha\beta} \to 0 \text{ or } \sigma_\alpha \to 0 \text{ as } r \to \infty. \qquad \text{...(3.3)}$$

The force-stresses and the couple-stresses admit the representation

$$\left.\begin{aligned} t_{11} &= \phi_{,22} - \psi_{,12},\ t_{22} = \phi_{,11} + \psi_{,12},\ t_{12} = -\phi_{,12} - \psi_{,22} \\ t_{21} &= -\phi_{,12} + \psi_{,11},\ m_{rz} = \sigma_1 = -\psi_{,1},\ m_{\theta z} = \sigma_2 = \psi_{,2} \end{aligned}\right\} \qquad \text{...(3.4)}$$

where ϕ and ψ are two arbitrary stress functions which satisfy the following differential equations

$$(\psi - c^2 \nabla^2 \psi)_{,1} = -2(1-\nu) b^2 \nabla^2 \phi_{,2} \qquad \text{...(3.5)}$$

and

$$(\psi - c^2 \nabla^2 \psi)_{,2} = 2(1-\nu) b^2 \nabla^2 \phi_{,1} \qquad \text{...(3.6)}$$

where ν denotes the Poisson's ratio. From eqns. (3.5) and (3.6) we get

$$\nabla^4 \phi = 0,\ \nabla^2(\psi - c^2 \nabla^2 \psi) = 0. \qquad \text{...(3.7)}$$

The displacement field u_i and the stress functions ϕ and ψ are related as

$$\left.\begin{aligned} u_{1,1} &= \frac{1}{2\mu}[\phi_{,22} - \psi_{,12} - \nu \nabla^2 \phi] \\ u_{2,2} &= \frac{1}{2\mu}[\phi_{,11} + \psi_{,12} - \nu \nabla^2 \phi] \\ u_{1,2} &= u_{2,1} = -\frac{1}{2\mu}[2\phi_{,12} - \psi_{,11} + \psi_{,22}]. \end{aligned}\right\} \qquad \text{...(3.8)}$$

we set

$$\hat{f}(s) = \int_{-\infty}^{\infty} f(x) \exp(is\,x)\,dx \equiv T\{f(x), s\} \qquad ...(3.9)$$

where s is the real transform parameter. With inversion theorem, equation (3.9) takes the form

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(s) \exp(-isx)\,ds. \qquad ...(3.10)$$

If

$$f^k(x) \equiv \frac{d^k f}{dx^k} \rightarrow 0 \text{ as } |x| \rightarrow \infty, (k = 0, 1, 2, \ldots N-1) \qquad ...(3.11)$$

then

$$T\{f^N(x), s\} = (-is)^N \hat{f}(s). \qquad ..(3.12)$$

Applying Fourier transform with respect to x_2 on the eqns. (3.5), (3.6), (3.7), and using (3.11), (3.12), we obtain

$$\frac{d}{dx_1}(\hat{\psi} - c^2 \hat{\nabla}^2 \hat{\psi}) = 2(1-\nu)\, b^2\, is\, \hat{\nabla}^2 \hat{\phi} \qquad ...(3.13)$$

$$-is(\hat{\psi} - c^2 \hat{\nabla}^2 \hat{\psi}) = 2(1-\nu)\, b^2 \frac{d}{dx_1}(\hat{\nabla}^2 \hat{\phi}) \qquad ...(3.14)$$

$$\hat{\nabla}^4 \hat{\phi} = 0, \; \hat{\nabla}^2 (\hat{\psi} - c^2 \hat{\nabla}^2 \hat{\psi}) = 0 \qquad ...(3.15)$$

where

$$\hat{\nabla}^2 = (D^2 - s^2), \text{ with } D \equiv \frac{d}{dx_1}.$$

With Fourier transform, eqns (3.4) and (3.8) become

$$\left.\begin{aligned}
&\hat{t}_{11}(x_1, s) = -s^2 \hat{\phi} + is \frac{d\hat{\psi}}{dx_1}, \; \hat{t}_{22}(x_1, s) = \frac{d^2 \hat{\phi}}{dx_1^2} - is \frac{d\hat{\psi}}{dx_1} \\
&\hat{t}_{12}(x_1, s) = is \frac{d\hat{\phi}}{dx_1} + s^2 \hat{\psi}, \; \hat{t}_{21}(x_1, s) = is \frac{d\hat{\phi}}{dx_1} + \frac{d^2 \hat{\psi}}{dx_1^2} \\
&\hat{\sigma}_1(x_1, s) = \frac{d\hat{\psi}}{dx_1}, \; \hat{\sigma}(x_1, s) = -is\,\hat{\psi}
\end{aligned}\right\} \qquad ...(3.16)$$

and

$$\left.\begin{aligned}\hat{u}_1(x_1, s) &= \frac{1}{2\mu s^2}\left\{(1-\nu)\frac{d^3\hat{\phi}}{dx_1^3} - (2-\nu)s^2\frac{d\hat{\phi}}{ds_1} + is^3\hat{\psi}\right\}\\ \hat{u}_2(x_1, s) &= \frac{1}{2\mu s}\left\{i(1-\nu)\frac{d^2\hat{\phi}}{dx_1^2} + i\nu s^2\hat{\phi} + s\frac{d\hat{\psi}}{dx_1}\right\}.\end{aligned}\right\} \quad \ldots(3.17)$$

The boundary and regularity conditions (3.1) to (3.3) can be written as

Case A :

$$\hat{t}_{11}(0, s), = -\hat{p}(s),\ \hat{t}_{12}(0, s) = \hat{\sigma}_1(0, s) = 0. \quad \ldots(3.18)$$

Case B :

$$\hat{t}_{12}(0, s) = \hat{p}(s),\ \hat{t}_{11}(0, s) = \hat{\sigma}_1(0, s) = 0. \quad \ldots(3.19)$$

Furthermore,

$$\hat{t}_{\alpha\beta}(x_1, s),\ \hat{\sigma}_\alpha(x_1, s) \to 0 \text{ as } x_1 \to \infty. \quad \ldots(3.20)$$

From

$\hat{\nabla}^4\hat{\phi} = 0$, we get $(D^2 - s^2)^2\hat{\phi} = 0$, which yields the solution

$$\hat{\phi}(x_1, s) = (A + Bx_1)\exp(-sx_1) \quad \ldots(3.21)$$

where $\exp(sx_1)$ has been omitted in view of (3.20). Similarly, $\nabla^2(\hat{\psi}\ \ c^2\hat{\nabla}^2\hat{\psi})=0$ gives

$$\hat{\psi}(x_1, s) = E\exp\left(-\frac{\sqrt{1+c^2s^2}}{c}x_1\right) + 4(1-\nu)b^2 isB\exp(-sx_1). \quad \ldots(3.22)$$

Here, A, B and E are arbitrary constants.

Substituting for $\hat{\phi}$ and $\hat{\psi}$ from the equations (3.21) to (3.22) into the eqn. (3.16), we obtain

$$\begin{aligned}\hat{t}_{11}(x_1, s) = -s^2(A+Bx_1)\exp(-sx_1) - \frac{is}{c}\{1+c^2s^2\}^{1/2}\\ \times E\exp\left\{-\frac{\sqrt{1+c^2s^2}}{c}x_1\right\}\\ + 4(1-\nu)b^2 is^2 B\exp(-sx_1)\end{aligned} \quad \ldots(3.23)$$

$$\begin{aligned}\hat{t}_{12}(x_1, s) = is(B-sA)\exp(-sx_1) + s^2E\exp\left\{-\frac{\sqrt{1+c^2s^2}}{c}x_1\right\}\\ + 4(1-\nu)b^2 is^3 B\exp(-sx_1)\end{aligned} \quad \ldots(3.24)$$

$$\hat{\sigma}_1 (x_1, s) = - E \exp \left\{ - \frac{\sqrt{1+c^2 s^2}}{c} x_1 \right\} - 4 (1-\nu) b^2 i s^2 B \exp (-s x_1). \quad \ldots(3.25)$$

The application of the boundary conditions now furnishes the values of constants A, B and E:

$$A = \frac{\hat{p}(s)}{s^2}, \quad B = \frac{\hat{p}(s)\sqrt{1+c^2 s^2}}{s\,[\sqrt{1+c^2 s^2} + 4(1-\nu) s^2 c^2 (b^2/c^2) \{\sqrt{1+c^2 s^2} - sc\}]}$$

$$E = - \frac{i\hat{p}(s)\, 4 (1-\nu) b^2 s^2 c}{s\,[\sqrt{1+c^2 s^2} + 4(1-\nu) s^2 c^2 (b^2/c^2) \{\sqrt{1+c^2 s^2} - sc\}]}. \quad \ldots(3.26)$$

The values of these constants can be simplified further with the help of eqn. (3.2):

$$A = 0, \quad B = \frac{\hat{p}(s)\, L(cs)}{is\, M(cs)}, \quad E = - \frac{4(1-\nu) b^2 s c \hat{p}(s)}{M(cs)} \quad \ldots(3.27)$$

where

$$L(s) = \sqrt{1+s^2}, \quad M(s) = L(s) + 4(1-\nu) s^2 (b^2/c^2) \{L(s) - s\}. \quad \ldots(3.28)$$

Substituting for these constants yields

Case A:

$$\left.\begin{aligned} \hat{\phi}(x_1, s) &= \frac{\hat{p}(s)}{s^2} \exp(-|s| x_1) + \frac{\hat{p}(s)\, L(cs)}{sM(cs)} x_1 \exp(-|s| x_1) \\ \hat{\psi}(x_1, s) &= \frac{4(1-\nu) b^2 i \hat{p}(s)}{M(cs)} \Big[- cs \exp\left\{ - \frac{L(cs)}{c} x_1 \right\} \\ &\quad + L(cs) \exp(-|s| x_1) \Big]. \end{aligned}\right\} \quad \ldots(3.29)$$

Case B:

$$\left.\begin{aligned} \hat{\phi}(x_1, s) &= - \frac{i\hat{p}(s)\, L(cs)}{sM(cs)} x_1 \exp(-|s| x_1) \\ \hat{\psi}(x_1, s) &= \frac{4(1-\nu) b^2 \hat{p}(s)}{M(cs)} \Big[- cs \exp\left\{ - \frac{L(cs)}{c} x_1 \right\} \\ &\quad + L(cs) \exp\{-s x_1\} \Big]. \end{aligned}\right\} \quad \ldots(3.30)$$

From here onwards, instead of $L(cs)$ and $M(cs)$, we shall simply write L and M, respectively. Using the inversion formula (3.10), the stresses, the couple stresses, and the displacements can be expressed as

$$t_{11}(x_1, x_2) = -\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(s^2\hat{\phi} - is\frac{d\hat{\psi}}{dx_1}\right)\exp(-isx_2)ds$$

$$t_{22}(x_1, x_2) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\frac{d^2\hat{\phi}}{dx_1^2} - is\frac{d\hat{\psi}}{dx_1}\right)\exp(-isx_2)\,ds$$

$$t_{12}(x_1, x_2) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(is\frac{d\hat{\phi}}{dx_1} + s^2\hat{\psi}\right)\exp(-isx_2)\,ds$$

$$t_{21}(x_1, x_2) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(is\frac{d\hat{\phi}}{dx_1} + \frac{d^2\hat{\psi}}{dx_1^2}\right)\exp(-isx_2)\,ds$$

$$\sigma_1(x_1, x_2) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{d\hat{\psi}}{dx_1}\exp(-isx_2)\,ds$$

$$\sigma_2(x_1, x_2) = \frac{-1}{2\pi}\int_{-\infty}^{+\infty} s\hat{\psi}\exp(-isx)\,ds \qquad ..(3.31)$$

and

$$u_1(x_1, x_2) = \frac{1}{4\pi\mu}\int_{-\infty}^{+\infty}\left\{(1-\nu)\frac{d^3\hat{\phi}}{dx_1^3} - 2(1-\nu)s^2\frac{d\hat{\phi}}{dx_1} + is^3\hat{\psi}\right\} \times\left(\frac{\exp(-isx_2)}{s^2}\right)ds \qquad ...(3.32)$$

$$u_2(x_1, x_2) = \frac{1}{4\pi\mu}\int_{-\infty}^{+\infty}\left\{i(1-\nu)\frac{d^2\hat{\phi}}{dx_1^2} + i\nu s^2\hat{\phi} + s\frac{d\hat{\psi}}{dx_1}\right\}\frac{\exp(-isx^2)}{s}\,ds. \qquad ...(3.33)$$

Writing $\hat{p}(s) = p_1(s) + ip_2(s)$, the formula (3.10) gives

$$\left.\begin{aligned} p_1(s) &= \int_{\infty}^{\infty}\{p(x) + p(-x)\}\cos sx\,dx \\ p_2(s) &= \int_{\infty}^{\infty}\{p(x) - p(-x)\}\sin sx\,dx. \end{aligned}\right\} \qquad ...(3.34)$$

Substitution for $\hat{\phi}$ and $\hat{\psi}$ from the equations (3.29) and (3.30) into the equations (3.31) to (3.33) leads to the solution for the force-stresses, the couple stresses, and the displacement field :

Case A :

$$t_{11}(x_1, x_2) = -\frac{1}{\pi}\int_0^{\infty}\Big[(M+Lsx_1)\, e^{-|s|x_1} + 4(1-\nu)\, b^2 Ls$$

$$\times \left(\exp\left(\frac{-Lx_1}{c}\right)\right) - \exp(-|s|x_1)\Big]\frac{p^-(x_2, s)}{M}\, ds$$

$$t_{22}(x_1, x_2) = \frac{1}{\pi}\int_0^{\infty}\Big[(M-2L+Lsx_1)\, e^{-|s|x_1} + 4(1-\nu)\, b^2 Ls^2$$

$$\times \left(\exp\left(\frac{-Lx_1}{c}\right)\right) - \exp(-sx_1)\Big]\frac{p^-(x_2, s)}{M}\, ds$$

$$t_{12}(x_1, x_2) = \frac{1}{\pi}\int_0^{\infty}\Big[(M-L+Lsx_1)\, e^{-sx_1} + 4(1-\nu)\, b^2 s^2$$

$$\times \left\{cs\exp\left(\frac{-Lx_1}{c}\right) - L\exp(-sx_1)\right\}\Big]\frac{p^-(x_2, s)}{M}\, ds$$

$$t_{21}(x_1, x_2) = \frac{-1}{\pi}\int_0^{\infty}\Big[(M-L+Lsx_1)\, e^{-sx_1} + 4(1-\nu)\, b^2 Ls$$

$$\times \left\{\frac{L}{c}\exp\left(\frac{-Lx_1}{c}\right) - s\exp(-sx_1)\right\}\Big]\frac{p^-(x_2, s)}{M}\, ds$$

$$\sigma_1(x_1, x_2) = \frac{4(1-\nu)\, b^2}{\pi}\int_0^{\infty}\left[\exp\left(-\frac{Lx_1}{c}\right) - \exp(-sx_1)\right]$$

$$\times \frac{Lsp^-(x_2, s)}{M}\, ds$$

$$\sigma_2(x_1, x_2) = \frac{-4(1-\nu)\, b^2}{\pi}\int_0^{\infty}\left[cs\exp\left(-\frac{Lx_1}{c}\right) - L\exp(-sx_1)\right]$$

$$\times \frac{sp^+(x_2, s)\, ds}{M} \qquad \text{...(3.35)}$$

$$u_1(x_1, x_2) = \frac{1}{2\pi\mu} \int_0^\infty \Big[(L+M-2L\nu+Lsx_1)\, e^{-sx_1} + 4(1-\nu)\, b^2 s^2$$

$$\times \left\{ cs \exp\left(\frac{-Lx_1}{c}\right) - L \exp(-sx_1) \right\} \Big] \frac{p^+(x_2, s)}{sM}\, ds$$

$$u_2(x_1, x_2) = \frac{1}{2\pi\mu} \int_0^\infty \Big[\{M-2(1-\nu)L+Lsx_1\}\, e^{-sx_1} + 4(1-\nu)$$

$$\times\ b^2 s^2 L \left\{ \exp\left(\frac{Lx_1}{c}\right) - \exp(-sx_1) \right\} \Big] \frac{(p^{-2} x_1, s)}{sM}\, ds \qquad \ldots(3.36)$$

where

$$\left.\begin{aligned} p^+(x_2, s) &\equiv p_1(s)\cos(sx_2) + p_2(s)\sin(sx_2) \\ p^-(x_2, s) &\equiv p_1(s)\sin(sx_2) - p_2(s)\cos(sx_2). \end{aligned}\right\} \qquad ..(3.37)$$

Case B :

$$t_{11}(x_1, x_2) = \frac{1}{\pi} \int_0^\infty \Big[sx_1\ e^{-sx_1} + 4(1-\nu)\, b^2 s^2 \left\{ \exp\left(\frac{-Lx_1}{c}\right) \right.$$

$$\left. - \exp(-sx_1) \right\} \Big] \frac{Lp^-(x_2, s)}{M}\, ds$$

$$t_{22}(x_1, x_2) = \frac{1}{\pi} \int_0^\infty \Big[(2-sx_1)\, e^{-sx_1} - 4(1-\nu)\, b^2 s^2 \left\{ \exp\left(\frac{-Lx_1}{c}\right) \right.$$

$$\left. - \exp(-sx_1) \right\} \Big] \frac{Lp^-(x_2, s)}{M}\, ds$$

$$t_{12}(x_1, x_2) = \frac{1}{\pi} \int_0^\infty \Big[L(1-sx_1)\ e^{-sx_1} - 4(1-\nu)\, b^2 s^2 \left\{ cs \exp\left(\frac{-Lx_1}{c}\right) \right.$$

$$\left. - L \exp(1 - sx_1) \right\} \Big] \frac{p^+(x_2, s)}{M}\, ds$$

$$t_{21}(x_1, x_2(= \frac{1}{\pi} \int_0^\infty \Big[(1-sx_1)\, e^{-sx_1} - 4(1-\nu)\, b^2 s \left\{ \exp\left(\frac{-Lx_1}{c}\right) \right.$$

$$\left. \times\ \frac{L}{c} - s \exp(-sx_1) \right\} \Big] \frac{Lp^+(x_2, s)}{M}\, ds$$

(*equation continued on p.* 1065)

$$\sigma_1(x_1, x_2) = \frac{4(1-\nu)b^2}{\pi} \int_0^\infty \left\{ \exp\left(\frac{-Lx_1}{c}\right) - \exp(-sx_1) \right\}$$

$$\times \frac{Lsp^+(x_2, s)}{M} ds$$

$$\sigma_2(x_1, x_2) = \frac{4(1-\nu)b^2}{\pi} \int_0^\infty \left\{ cs \exp\left(\frac{-Lx_1}{c}\right) - L \exp(-sx_1) \right.$$

$$\times \frac{sp^-(x_2, s)}{M} ds \quad \ldots(3.38)$$

and

$$u_1(x_1, x_2) = \frac{1}{2\pi\mu} \int_0^\infty \left\{ (4b^2 s^2 - sx_1 - 1)(1-\nu) Le^{-sx_1} - 4b^2 cs^3 \right.$$

$$\times (1-\nu) \exp\left(\frac{-Lx_1}{c}\right)\Bigg\} \frac{p(x_2, s)}{SM} ds$$

$$u_2(x_1, x_2) = \frac{1}{2\pi\mu} \int_0^\infty \Big[\{(1-\nu)(sx_1 - 2 - 4b^2 s^2) + \nu s\} e^{-sx_1}$$

$$+ 4(1-\nu) b^2 s^2 \exp\left(\frac{-Lx_1}{c}\right)\Big] \frac{Lp^-(x_2, s)}{SM} ds. \quad \ldots(3.39)$$

4. Half Plane Under Concentrated Normal and Tangential Loads

We plan to examine the stress concentration due to concentrated normal and tangential load applied at the origin. Once more, we take the bounding edge of the half plane to be free from couple tractions. The results ensuing from this type of loading can be obtained by the usual limit process from the results of section 3. We assume that the load is applied to a small finite segment 2ϵ of the edge and proceed to the limit as ϵ approaches zero. If P denotes the scalar value of the given concentrated load applied at the origin, the load function p of the last section can be characterized as

$$p_\epsilon(x_2) = 0, \; (\epsilon < x_2 < \infty) \quad \ldots(4.1)$$

$$p_\epsilon(x_2) \geqslant 0, \; (0 \leqslant |x_2| < \epsilon) \quad \ldots(4.2)$$

$$\int_{-\epsilon}^{\epsilon} p_\epsilon(x_2)\, dx_1 = P. \quad \ldots(4.3)$$

The auxiliary functions $p^{+}(x_2, s)$ and $p^{-}(x_2, s)$ occurring earlier in (3.37) have to be replaced by $p_{\epsilon}^{+}(x_2, s)$ and $p_{\epsilon}^{-}(x_2, s)$, respectively.

We write

$$p_{\epsilon}^{+}(x_2, s) = P\cos(sx_2),\ p_{\epsilon}^{-}(x_2, s) = P\sin(sx_2). \quad \ldots(4.4)$$

The desired stresses as well as couple stresses can now be obtained by inserting from eqns. (4.4) into eqns. (3.35) and taking the limit as $\epsilon \to 0$. Let the open half plane D_1 together with its bounding edge be denoted by $\overline{D}_1$. The improper integrals in eqns. (3.35) and (3.38) with value of $p_{2}^{+}(x_2, s)$ and $p_{2}^{-}(x, s)$ as in (4.4) are convergent throughout $\overline{D}_1$, except at the origin. When $x_\alpha x_\alpha \to 0$, most of these integrals become divergent, and the force-stresses and couple-stresses accordingly become unbounded at the point of application of the load. It is our intention to seek out all such singularities in the closed form. To that end, we use the notation

$$\left.\begin{aligned} t_{\alpha\beta}(x_1, x_2) &= \int_0^\infty \hat{t}_{\alpha\beta}(x_1, x_2; s)\, ds \\ \sigma_\alpha(x_1, x_2) &= \int_0^\infty \hat{\sigma}_\alpha(x_1, x_2; s)\, ds \end{aligned}\right\} \quad \ldots(4.5)$$

where $\hat{t}_{\alpha\beta}$ and $\hat{\sigma}_\alpha$ stand for the integrands after absorption of all constant multipliers belonging to $t_{\alpha\beta}$ and σ_α. Since $\hat{t}_{\alpha\beta}(x_1, x_2; s)$ and $\hat{\sigma}_\alpha(x_1, x_2; s)$ are finite and continuous on $\overline{D}_1$ for every fixed $s \geqslant 0$ in view of eqns. (3.35), (3.38), (4.4), and (4.5), it is clear that any possible divergence at the origin of the integrals in (4.5) must stem from the behaviour of $\hat{t}_{\alpha\beta}(0, 0; s)$ and $\hat{\sigma}_\alpha(0, 0; s)$ as $s \to \infty$. Specifically, the singularities in question must be contributed by those portions of the integrals that are of the order $O(s^{-1})$ or of a larger order of magnitude as $s \to \infty$ at the point $x_1 = x_2 = 0$. The parts of $\hat{t}_{\alpha\beta}(x_1, x_2; s)$ and $\hat{\sigma}_\alpha(x_1, x_2; s)$ which dominate the behaviour of these functions $L(cs)$ and $M(cs)$. Thus, for large value of s, we have

$$L = cs + \frac{1}{2cs} - \frac{1}{8c^3s^3} + O(s^{-5})$$

$$M = \{1+2(1-\nu)b^2/c^2\}\, cs + \{1-2(1-\nu)(b^2/c^2)(1/2\, cs) + O(s^{-3})\}$$

$$\exp\left(\frac{-Lx_1}{c}\right) = \exp(-sx_1)\left[1 - sx_1\left\{\frac{1}{2c^2s^2} - \frac{1}{8c^4s^4} + O(s^{-6})\right\}\right]$$

$$+ \frac{1}{2}s^2x_1^2\left\{\frac{1}{2c^2s^2} - \frac{1}{8c^4s^4} + O(s^{-6})\right\} \quad c > 0 \quad x_1 \geqslant 0. \quad \ldots(4.6)$$

As the method adopted for all components of $t_{\alpha\beta}$ and σ_α in both the loading cases is the same, it is sufficient to discuss one component in detail, say t_{11} in Case A. From eqns. (3.35), (4.4) and (4.5), we have

$$\hat{t}_{11}(x_1, x_2; s) = -(P/\pi)\exp(-sx_1)\cos(sx_2)\left[\frac{1-2(1-\nu)(b^2/c^2)}{1+2(1-\nu)(b^2/c^2)}x_1 s + 1 + \frac{(1-\nu)(b^2/c^2)(x_1^2/c^2)}{2\{1+2(1-\nu)(b^2/c^2)\}} + \frac{2\{(1-\nu)(b^2/c^2)\}\{1-(1-\nu)(b^2/c^2)\}(x_1/c)}{[1+2(1-\nu)b^2/c^2)]^2}(cs) - \frac{1}{12}\left(\frac{x_1^3}{c^3}\right)\frac{1}{cs}\cdot\frac{(1-\nu)(b^2/c^2)}{1+2(1-\nu)(b^2/c^2)}\right]. \quad ...(4.7)$$

Use of eqns. (4.5) and (4.7) in eqns. (3.35) yields

$$t_{11}(x_1, x_2) = \frac{-P}{\pi}\int_0^\infty\left[\frac{1-2(1-\nu)(b^2/c^2)}{1+2(1-\nu)(b^2/c^2)}x_1 s + 1 + \frac{(1-\nu)(b^2/c^2)(x_1^2/c^2)}{2\{1+2(1-\nu)(b^2/c^2)\}} + \frac{(1-\nu)(b^2/c^2)}{1+2(1-\nu)(b^2/c^2)} \times \left\{\frac{2\{1-(1-\nu)(b^2/c^2)\}(x_1/c)}{1+2(1-\nu)(b^2/c^2)} - \frac{1}{12}\frac{x_1^3}{c^3}\right\}\frac{1}{cs}\right] \times \{\exp(-sx_1)\cos(sx_2)\}(ds). \quad ...(4.8)$$

With the help of Fourier integral transforms (Sneddon 1951), and regarding the stresses to be unbounded at $r = 0$, eqn. (4.8) gives

$$t_{11}(x_1, x_2) = \frac{-2Px_1\{x_1^2(1-\nu)(b^2/c^2)x_2^2\}}{\pi\{1+2(1-\nu)b^2/c^2)\}r^4} + O(1). \quad ...(4.9)$$

Equation (4.9) is valid for all (x_1, x_2) in $\bar{D}_1$, except at the origin, and for every $b, c > 0$ as $r \to 0$.

The same procedure can be applied to other stress components for both cases of loading. We write below their derivations:

Case A :

$$t_{11}(x_1, x_2) = \left(\frac{2Px_1}{\pi r^4}\right)\frac{\{x_1^2 + 2(1-\nu)(b^2/c^2)x_2^2\}}{\{1+2(1-\nu)(b^2/c^2)\}} + O(1)$$

$$t_{22}(x_1, x_2) = \left(\frac{2Px_1x_2^2}{\pi r^4}\right)\frac{\{2(1-\nu)(b^2/c^2)-1\}}{1+2(1-\nu)(b^2/c^2)} + O(1)$$

$$t_{12}(x_1, x_2) = \left(\frac{2Px_1^2x_1}{\pi r^4}\right)\frac{\{2(1-\nu)(b^2/c^2)-1\}}{1+2(1-\nu)(b^2/c^2)} + O(1)$$

(*equation continued on p.* 1068)

$$t_{21}(x_1, x_2) = \left(\frac{-2Px_2}{\pi r^4}\right) \frac{\{x_1^2 + 2(1-\nu)(b^2/c^2)x_2^2\}}{\{1+2(1-\nu)(b^2/c^2)\}} + O(1)$$

$$\sigma_1(x_1, x_2) = O(1)$$

$$\sigma_2(x_1, x_2) = \frac{-2p}{\pi} \frac{(1-\nu)(b^2/c^2)}{\{1+2(1-\nu)(b^2/c^2)\}} \log r + O(1). \qquad \ldots(4.10)$$

Case B :

$$t_{11}(x_1, x_2) = \left(\frac{-2Px_1^2 x_2}{\pi r^4}\right) \left\{\frac{2(1-\nu)(b^2/c^2)-1}{1+2(1-\nu)(b^2/c^2)}\right\} + O(1)$$

$$t_{22}(x_1, x_2) = \left(\frac{2Px_2}{\pi r^4}\right) \left\{\frac{2(1-\nu)(b^2/c^2)x_1^2 + x_2^2}{1+2(1-\nu)(b^2/c^2)}\right\} + O(1)$$

$$t_{12}(x_1, x_2) = \left(\frac{2Px_1}{\pi r^4}\right) \left\{\frac{2(1-\nu)(b^2/c^2)x_1^2 + x_2^2}{1+2(1-\nu)(b^2/c^2)}\right\} + O(1)$$

$$t_{21}(x_1, x_2) = \left(-\frac{2Px_1 x_2^2}{\pi r^4}\right) \left\{\frac{2(1-\nu)(b^2/c^2) - 1}{1+2(1-\nu)(b^2/c^2)}\right\} + O(1)$$

$$\sigma_1(x_1, x_2) = \sigma_2(x_1, x_2) = O(1). \qquad \ldots(4.11)$$

where the functions $O(1)$ are finite and continuous throughout $\bar{D}_1$, though not necessarily analytic at the origin. These bounded terms do not, of course, contribute to the stress resultant of the singularity at the origin.

Letting both b and c tend to zero in equations (3.35) and (3.38), one obtains the corresponding results in the classical elasticity for both cases of loading :

Case A :

$$t_{11}^0(x_1, x_2) = \frac{-2Px_1^3}{\pi r^4}, \; t_{22}^0(x_1, x_2) = \frac{-2Px_1 x_2^2}{\pi r^4},$$

$$t_{12}^0(x_1, x_2) = t_{21}^0(x_1, x_2) = \frac{-2Px_1^2 x_2}{\pi r^4}, \; \sigma_1^0(x_1, x_2)$$

$$= \sigma_2^0(x_1, x_2) = 0. \qquad \ldots(4.12)$$

Case B :

$$t_{11}^0(x_1, x_2) = \frac{2Px_1^2 x_2}{\pi r^4}, \; t_{22}^0(x_1, x_2) = \frac{2Px_2^3}{\pi r^4},$$

$$t_{12}^0(x_1, x_2) = t_{21}^0(x_1, x_2) = \frac{2Px_1 x_2^2}{\pi r^4}, \; \sigma_1^0(x_1, x_2)$$

$$= \sigma_2^0(x_1, x_2) = 0. \qquad \ldots(4.13)$$

On comparison of results in equations (4.10) and (4.11) with those in equations (4.12) and (4.13) reveals that the singularities at the origin inherent in $t_{\alpha\beta}$, $(b > c > 0)$ and $t^0_{\alpha\beta}$ are of the same order $O(r^{-1})$. The singular terms in (4.10) and (4.11) indicate that there is a strong dependence of the strength of singularity on microrotation through a new ratio (b/c), which, however is not the case in classical elasticity.

5. Stress Concentration Due to Discontinuous Tangential Loads

We now discuss the plane strain problem of half plane for uniformly distributed shearing tractions applied over a finite segment of the bounding edge. The entire boundary is free from normal and couple tractions.

For the load function P, we take the following choice :

$$\left.\begin{aligned} p(x_2) &= p_0, \ 0 < |x_2| < a \\ p(x_2) &= 0, \ 0 < |x_2| < \infty \end{aligned}\right\} \qquad \ldots(5.1)$$

where p_0 is a constant, and $2a$ denotes the length of the load segment.

From eqns. (3.9) and (5.1),

$$\hat{p}(s) = 2\int_0^a p_0 (\cos sx + i \sin sx)\, dx \qquad \ldots(5.2)$$

so that the real part of the load segment is $2p \sin (sa)/s$. This, combined with eqns. (4.4) yields

$$\left.\begin{aligned} p^+(x_2, s) &= \left(\frac{1}{s}\right) 2p_0 \sin (sa) \cos (sx_2) \\ p^-(x_2, s) &= \left(\frac{1}{s}\right) 2p_0 \sin (sa) \sin (sx_2). \end{aligned}\right\} \qquad \ldots(5.3)$$

The use of eqns. (5.3) in eqns. (3.38) and (3.39) leads to the following results :

$$t_{11}(x_1, x_2) = \frac{2p_0}{\pi} \int_0^\infty \Big[sx_1\, e^{-sx_1} + 4(1-\nu)\, b^2 s^2$$

$$\times \left\{ \exp\left(\frac{-Lx_1}{c} - e^{-sx_1}\right)\right\}\Big] \frac{L \sin (sa) \sin (sx_2)}{sM}\, ds$$

$$t_{22}(x_1, x_2) = \frac{2p_0}{\pi} \int_0^\infty \Big[(2 - sx_1)\, e^{-sx_1} - 4(1-\nu)\, b^2 s^2$$

(*equation continued on p.* 1070)

$$\times \left\{\exp\left(\frac{-Lx_1}{c}\right) - e^{-sx_1}\right\}\Bigg] \frac{L \sin(sa) \sin(sx_2)}{sM}\, ds$$

$$t_{12}(x_1, x_2) = \frac{2p_0}{\pi} \int_0^\infty \Bigg[L(1-sx_1)\, e^{-sx_1} - 4(1-\nu)\, b^2 s^2$$

$$\times \left\{\exp\left(\frac{-Lx_1}{c}\right) cs - Le^{-sx_1}\right\}\Bigg] \frac{\sin(sa)\cos(sx_2)}{sM}\, ds$$

$$t_{21}(x_1, x_2) = \frac{2p_0}{\pi} \int_0^\infty [1 - sx_1)\, e^{-sx_1} - 4(1-\nu)\, b^2 s$$

$$\times \left\{\frac{L}{c} \exp\left(\frac{-Lx_1}{c}\right) - se^{-sx_1}\right\} \frac{L \sin(sa)\cos(sx_2)}{sM}\, ds$$

$$\sigma_1(x_1, x_2) = \frac{2p_0}{\pi}\, 4(1-\nu)\, b^2 \int_0^\infty \left\{\exp\left(\frac{-Lx_1}{c}\right) - e^{-sx_1}\right\}$$

$$\times \left(\frac{L}{M}\right) (\sin sa)(\cos sx_2)\, ds$$

$$\sigma_2(x_1, x_2) = \frac{2p_0}{\pi}\, 4(1-\nu)\, b^2 \int_0^\infty \left\{cs \exp\left(\frac{-Lx_1}{c}\right) - Le^{-sx_1}\right\}$$

$$\times \left(\frac{1}{M}\right) (\sin sa)\ (\cos sx_2)\, ds \qquad \ldots(5.4)$$

and

$$u_1(x_1, x_2) = \frac{p_0}{\pi\mu} \int_0^\infty \Bigg\{(4b^2 s^2 - sx_1 - 1)(1-\nu)\, Le^{-sx_1} - 4b^2 cs^3 (1-\nu)$$

$$\times \exp\left(\frac{-Lx_1}{c}\right)\Bigg\} \frac{\sin(sa)\sin(sx_2)}{s^2 M}\, ds$$

$$u_2(x_1, x_2) = \frac{p_0}{\pi\mu} \int_0^\infty \Bigg[\{(1-\nu)(sx_1 - 2 - 4b^2 s^2) + \nu s\}\, e^{-sx_1} + 4(1-\nu)$$

$$\times b^2 s^2 \exp\left(\frac{-Lx_1}{c}\right)\Bigg] \frac{L \sin(sa)\cos(x_2 s)}{Ms^2}\, ds. \qquad \ldots(5.5)$$

On examining the above solution, we note that all the improper integrals in it are convergent throughout the half plane $\overline{D_1}$ with the possible exception of the end points

Q_1 (0, a) and Q_2 (0, $-a$) of the load interval. The technique employed in section 4 can also be applied in this case to derive the stress and the couple stress fields in the neighbourhood of the points Q_α, $\alpha = 1, 2$:

$$t_{11}(x_1, x_2) = O(1)$$

$$t_{22}(x_1, x_2) = \frac{2p_0 \log(r_2/r_1)}{\pi\,[1+2(1+\nu)(b^2/c^2)]} + O(1)$$

$$t_{12}(x_1, x_2) = t_{21}(x_1, x_2) = O(1)$$

$$\sigma_\alpha(x_1, x_2) = O(1),\ \alpha = 1, 2 \qquad \ldots(5.6)$$

where

$$r_1^2 = x_1^2 + (x_2 - a)^2,\ r_2^2 = x_1^2 + (x_2 + a)^2.$$

We observe that there is only one singularity in the stress field for discontinuous tangential loading i.e. $t_{22}(x_1, x_2)$. Letting $b = c = 0$, $L = M = 1$, in eqns. (5.4) and (5.5), we recover the classical solution which after simplification, assumes the form

$$t_{11}(x_1, x_2) = \frac{-P_0}{2\pi}\{\cos 2\theta_1 - \cos 2\theta_2\}$$

$$\bar{t}_{22}(x_1, x_2) = \frac{p_0}{2\pi}\{4\log(r_2/r_1) + \cos 2\theta_1 - \cos 2\theta_2\}$$

$$\bar{t}_{12}(x_1, x_2) = \bar{t}_{21}(x_1, x_2) = \frac{p_0}{2\pi}\{(2\theta_1 - 2\theta_2) + \sin 2\theta_1 - \sin 2\theta_2\}$$

$$\bar{\sigma}_\alpha(x_1, x_2) = 0 \qquad \ldots(5.7)$$

where θ_1 and θ_2 are the polar angles (shown in Figure 1) given by

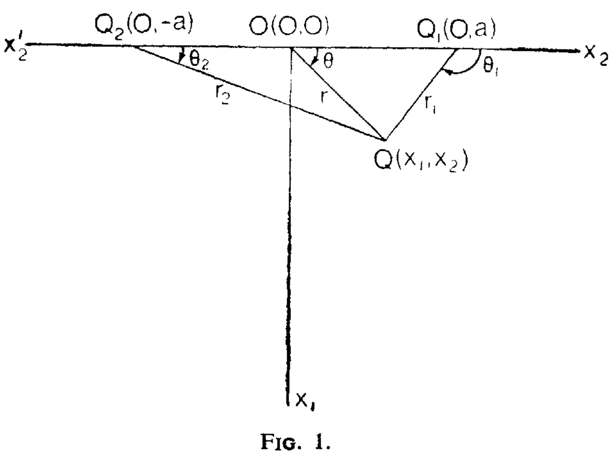

FIG. 1.

$$\theta_1 = \tan^{-1} \frac{x_1}{x_2 - a}, \; \theta_2 = \tan^{-1} \frac{x_1}{x_2 + a}, \; (0 \leqslant \theta_\alpha \leqslant \pi).$$

In order to gain a quantitative insight into the influence exerted by the couple stresses based on the micropolar model upon the singular stress t_{22}, it would be appropriate to compute the numerical value of $t_{22}(0, x_2)$ for $-a < x_2 < a$.

From eqns. (5.4),

$$t_{22}(0, x_2) = \frac{4p_0}{\pi} \int_0^\infty \frac{L(cs)}{M(cs)} \frac{1}{s} \sin(sa) \sin(sx_2)\, ds \qquad \ldots(5.8)$$

which is valid for all x_2 except $x_2 = \pm a$. The convergence of the improper integral in (5.8) when $| x_2 | \neq a$ is assured by "Dirichlet's test" (Bromwich 1949). Substituting for $L(cs)$ and $M(cs)$ from eqns. (4.6) into the eqns. (5.8), we get

$$\frac{\pi}{2p_0} t_{22}(0, x_2) = \int_0^1 \frac{1/s \sqrt{1 + s^2} \,.\, 2 \sin(sa/c) \sin(sx_2/c)}{\sqrt{1 + s^2} + 4(1-\nu)(b^2/c^2) s^2 \{\sqrt{1 + s^2} - s\}} ds$$

$$+ \int_1^\infty \left[\frac{1/s}{1 + 2(1-\nu)(b^2/c^2)} + \frac{3/2\,(1-\nu)(b^2/c^2)}{\{1+2(1-\nu)\, b^2/c^2)\}^2} \right.$$

$$+ \frac{1}{(t^2+s^2)\frac{3}{2}} + \frac{(1-\nu)(b^2/c^2)}{[288\{1+2(1-\nu(b^2/c^2)\}^4]}$$

$$\left. \times \{65 + 80(1-\nu)(b^2/c^2) + 62(1-\nu)^2 (b^4/c^4)\} \frac{1}{s^7} \right]$$

$$\times \frac{2}{s} \sin\left(\frac{sa}{c}\right) \sin\left(\frac{sx_2}{c}\right) ds \qquad \ldots(5.9)$$

where

$$t^2 = \frac{5 + (1-\nu)(b^2/c^2)}{9\{1+2(1-\nu)(b^2/c^2)\}}. \qquad \ldots(5.10)$$

We choose $\nu = 0.5$, $(b/c) = 0.2$, $(a/c) = 1$, $t = 0.7323$, and use the integral formulas

$$\int_1^\infty \frac{1}{s} \sin(sa) \sin(sx_2)\, ds = \frac{1}{2} \log \left| \frac{x_2 + a}{-x_2 + a} \right|, \; a > 0, \; a \neq x_2 \qquad \ldots(5.11)$$

$$\int_1^\infty \frac{\cos(gt)}{(t^2+s^2)^{3/2}} ds = g/t\, K_1(gt), \; g > 0, \; t > 0 \qquad \ldots(512)$$

where K_1 is the modified Bessel's function of second kind and order one.

Thus, we have

$$\frac{\pi}{2p_0}\, t_{22}(0, x_2) = \int_0^1 \left\{\frac{1/s\ \sqrt{1+s^2}\ \sin 2\,(s)\sin(sx_2/a)}{\sqrt{1+s^2} + 2\,(0.04)\, s^2\,\{\sqrt{1+s^2}-s\}}\right\} ds$$

$$+ \left\{(0.9615)\log\left|\frac{a+x_2}{a-x_2}\right| + (0.03787)\left[\left(1-\frac{x_2}{a}\right)^2 K_1 t\right.\right.$$

$$\left.\left. - K_1\left(1+\frac{x_2}{a}\right)^2 t\right]\right\} + 0.00395\int_1^{\infty}\frac{1}{s^7}\, 2\sin(s)$$

$$\times \sin\left(\frac{sx_2}{c}\right) ds. \qquad \ldots(5.13)$$

We apply Filon's numerical scheme to evaluate the integral in eqn. (5.13). The numerical values of the stress component t_{22} along the boundary $x_1 = 0$ are tabulated in Table I, and then compared in Table II with the corresponding values $\bar{t}_{22}$ of the classical elasticity. The tabulated values are sketched in Figs. 2 and 3.

TABLE I

x_2/a	0	0.2	0.4	0.6	0.8
$\frac{\pi}{2p_0}\, t_{22}(0, x_2)$	0	0.3833	0.7734	1.1963	1.9887

TABLE II

x_2/a	0.2	0.4	0.6	0.8
$\frac{t_{22}(0, x_2)}{\bar{t}_{22}(0, x_2)}$	1.0883	1.0608	1.0492	1.0421

Figure 2 illustrates the variation of the dimensionless boundary values $\frac{\pi}{2p_0}\, t_{22}\ (0, x_2)$ with (x_2/a). The influence of couple stresses upon the boundary values of t_{22}, especially in the vicinity of the singular point $x_2 = a$, is brought out more clearly in Figure 3. It is apparent that the stresses $t_{\alpha\beta}$, with the exception of t_{22}, remain bounded in our theory as also in other theories.

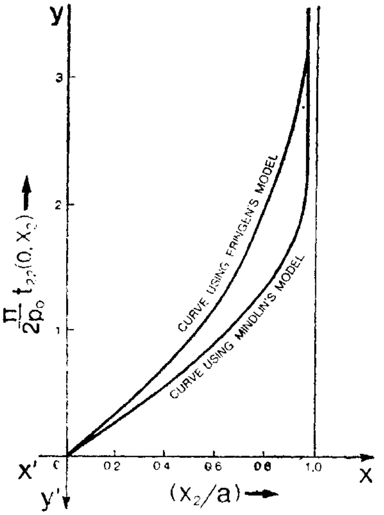

FIG. 2

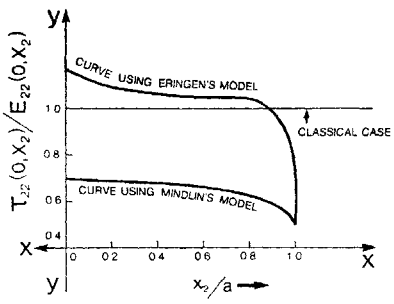

FIG. 3

It may be noted that both the couple stresses σ_1 and σ_2 remain finite at the points Q_1 and Q_2. The order of the singularity in t_{22} in our micropolar theory remains the same as in the classical theory, but the strength of the singularity depends not only on ν but also on the new ratio (b/c). If we take $b/c = 1$, we recover the results of the classical couple stress theory.

ACKNOWLEDGEMENT

The author wishes to thank Professor P. D. S. Verma for his kindness in making valuable suggestions. This paper includes research efforts supported by NSERC under grant no.

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Corrigendum

ON SOME RECENT RESULTS ON COMMON FIXED POINTS *by* S. L. SINGH

(*Indian J. Pure Appl. Math.*, 13(7), 757–61)

In fact, a slight alteration only in conditions (1.2) of Theorems 1–2, (3.2) of Theorems 3–4 and (3.2′) of Remark 2 is essential. The sup condition in (1.2), (3.2) (3.2′) should be replaced by

$$\sup \{d(TSx_i, TSx_j) : i, j \in N \text{ and } i, j \text{ are not of the same parity}\}$$

$$= \sup \{d(TSx_i, TSx_j) : i, j \in N\} < \infty,$$

$$\sup \{d(QPx_i, QPx_j) : i, j \in \omega \text{ and } i, j \text{ are not of the same parity}\}$$

$$= \sup \{d(QPx_i, QPx_j) : i, j \in \omega\} < \infty,$$

and

$$\sup \{d(Px_i, Px_j) : i, j \in \omega \text{ and } i, j \text{ are not of the same parity}\}$$

$$= \sup \{d(Px_i, Px_j) : i, j \in \omega\} < \infty, \text{ respectively.}$$

(By i, j are not of the same parity we mean if one of them is odd then the other is even.) These alterations cause no change, whatsoever, in the paper.

The author wishes to thank Dr. Do Hong Tan for the letter, Dr. J. H. M. Whitfield of Lakehead University for discussing the above alterations.