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FORCED VIBRATION OF AN ANISOTROPIC DIAPHRAGM WITH A CONCENTRIC HOLE

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In this problem the authors discuss the forced harmonic vibrations of a circular diaphragm of a homogeneous anisotropic elastic material of a particular type (reinforced concrete, Timoshenko and Wionowsky-Krieger 1959) with a concentric, hole in response to the load field $q(r)\, e^{ipt}$. The technique of finite Hankel transform is applied and behaviour of displacement field is analysed for $p = 1$ and $p = \pi$ for the different values of t when $\bar{q} = 0.713391;\ 1.5808828;\ 3.989238$ where $\bar{q}$ is Hankel transform of $q(r) = q_0\, r^{m-1}$ for $m = 1, 3, 5$. It is noted that for $p = 1$ in the initial stage the displacement profiles are sinusoidal but for large values of t they tend to become unimodal. Moreover, the displacements are comparatively larger near the inner edge. But when $p = \pi$, the displacements are larger near the outer edge and smaller near the inner edge. The case of isotropic medium is also compared with that of anisotropic medium graphically.

1. INTRODUCTION

Turner (1909) found the yield pressure on the internal boundary of a homogeneous isotropic cylindrical shell and obtained the pressure on the internal boundary required to produce a plastic flow up to a certain distance from the axis of the cylinder. From that time some investigations have been made on the effects of non-homogeneity on elastico-plastic stresses in a cylindrical shell. Ghosh (1963) worked on elastico-plastic stresses in a spherical pressure vessel of non-homogeneous material. His (Ghosh 1968) investigations on elastico-plastic behaviour are also of considerable interest.

The problem concerning the forced vibration of an isotropic plate with various conditions has been discussed by numerous investigators. The vibration of a

condenser microphone with a plate diaphragm of isotropic material under the action of an external periodic force is also well known. Periods of torsional vibrations of a homogeneous solid cylinder and a slab with a cylindrical hole are known (Love 1944). De (1971, 1973) has also investigated problems of this class.

In this paper the authors have investigated the forced harmonic vibrations of a circular diaphragm with a concentric hole, of an isotropic material of a particular type (reinforce concrete) and the behaviour of the displacement field in response to the load field is analysed.

2. Formulation of the Problem and its Solution

The differential equation of motion for an isotropic plate under investigation can be written (Timoshenko and Woinowsky-Krieger 1959) as

$$D_1 \frac{\partial^4 w'}{\partial x'^4} + 2H \frac{\partial^4 w'}{\partial x'^2 \partial y'^2} + D_2 \frac{\partial^4 w'}{\partial y'^4} + \rho' h' \frac{\partial^2 w'}{\partial t'^2} = Q'(x', y', t') \qquad \ldots(2.1)$$

where w' is displacement at any point,

D_1, D_2 are the elastic constants of the material,

h' = thickness of the plate (diaphragm),

ρ' = density of the material of the plate (diaphragm).

$H = \sqrt{D_1 D_2}$

and $Q'(x', y', t')$ denotes the intensity of the force at any time t'.

Following De (1971, 1973) and substituting

$$x_1 = x' \; ; \quad y_1 = y' (D_1/D_2)^{1/4}$$

and

$$x_1 = r' \cos\theta' \; ; \; y_1 = r' \sin\theta'$$

we obtain from (2.1)

$$D_1 \left(\frac{\partial^2}{\partial r'^2} + \frac{1}{r'} \frac{\partial}{\partial r'} \right)^2 w' + \rho' h' \frac{\partial^2 w'}{\partial t'^2} = Q' \qquad \ldots(2.2)$$

where, due to symmetry, Q' and w' are independent of θ'.

Now we solve eqn. (2.2) with the conditions,

$$\left.\begin{aligned} & t' = 0, \; w' = 0, \frac{\partial w'}{\partial t'} = 0 \text{ everywhere} \\ & t' = 0, \; w' = 0, \frac{\partial^2 w'}{\partial r'^2} + \frac{\nu}{r'} \frac{\partial^2 w'}{\partial r'} = 0 \text{ for } r' = a \text{ and } r' = b \end{aligned}\right\} \quad \ldots(2.3)$$

where ν is Poisson's ratio and a, b are radii of the plate ($a < b$). The second condition of (2.3) can be written as (Nowacki 1963)

$$\frac{\partial^2 w'}{\partial r'^2} + \frac{1}{r'} \frac{\partial w'}{\partial r'} = 0 \qquad \ldots(2.3a)$$

without any appreciable error in the final result.

Considering

$$Q'(r', t') = q'(r') e^{ip't'} \qquad \ldots(2.4)$$

where $q'(r')$ has dimensions of force per unit area and introducing the non-dimensional quantities

$$w = \frac{w'}{h'}, \quad r = \frac{r'}{a}, \quad t = \frac{t'}{\sqrt{\rho' h' a^4/D_1}}$$

$$q = \frac{a^4 q'}{h' D_1'}, \quad p = \sqrt{\frac{\rho' h' a^4}{D_1}}\, p' \qquad \ldots(2\ 5)$$

where D_1 has dimensions of $E h'^3$, E being the Young's modulus eqn. (2.2) reduces to

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r}\right]^2 w + \frac{\partial^2 w}{\partial t^2} = q(r) e^{ipt}. \qquad \ldots(2.6)$$

We now apply the finite Hankel Transforms (Sneddon 1951), as follows :

$$\bar{\omega} = \int_1^{r_1} rw \left[J_0(r\alpha)\, Y_0(r_1\alpha) - J_0(r_1\alpha)\, Y_0(r\alpha)\right] dr$$

$$\bar{q}(\alpha_n) = \int_1^{r_1} rq \left[J_0(r\alpha)\, Y_0(r_1\alpha) - J_0(r_1\alpha)\, Y_0(r\alpha)\right] dr \qquad \ldots(2.7)$$

where α is the root of the equation

$$J_0(\alpha_n)\, Y_0(\alpha_n r_1) - J_0(\alpha_n r_1)\, Y_0(\alpha_n) = 0$$

$n = 1, 2, 3 \ldots$ and $r_1 = b/a$.

Consequently eqn. (2.6) is transformed to

$$\alpha_n^4 \bar{w} + \frac{d^2 \bar{w}}{dt^2} = \bar{q} e^{ipt}. \qquad \ldots(2.8)$$

Its solution is

$$\bar{w} = A \cos \alpha_n^2 t + B \sin \alpha_n^2 t + \bar{q} e^{ipt} \Big/ (\alpha_n^4 - p^2) \qquad \ldots(2.9)$$

provided $p^2 \neq \alpha_n^4$.

Now applying the transformed initial conditions on equation (2.9), we get the values of A and B as follows :

$$A = -\frac{\bar{q}}{\alpha_n^4 - p^2}\ ;\ B = -\frac{ip\bar{q}}{\alpha_n^2(\alpha_n^4 - p^2)}\ .$$

Therefore the solution becomes

$$\bar{w} = -\frac{\bar{q}}{(\alpha_n^4 - p^2)}\cos\alpha_n^2 t - \frac{ip\bar{q}}{\alpha_n^2(\alpha_n^4 - p^2)}\sin\alpha_n^2 t + \frac{\bar{q}e^{ipt}}{(\alpha_n^4 - p^2)}\ . \qquad \ldots(2.10)$$

Inverse Hankel Transform of $\bar{w}$ is given by

$$w(r) = \Sigma \frac{2\alpha_n^2 J_0^2(\alpha_n r_1)\left[-\frac{\cos\alpha_n^2 t}{(\alpha_n^4 - p^2)} - \frac{ip\sin\alpha_n^2 t}{\alpha_n^2(\alpha_n^4 - p^2)} + \frac{e^{ipt}}{(\alpha_n^4 - p^2)}\right]\bar{q}}{J_0^2(\alpha_n) - J_0^2(\alpha_n r_1)}$$
$$\times [J_0(\alpha_n r)\, Y_0(\alpha_n r_1) - J_0(\alpha_n r_1)\, Y_0(\alpha_n r). \qquad \ldots(2.11)$$

This gives displacement field depending upon the form of load q.

Now taking $q(r) = q_0 r^{m-1}$, we have

$$\begin{aligned}\bar{q}(\alpha_n) = q_{0n}^{m+1} [&Y_0(\alpha_n r_1)\{((m-1)\alpha_n r_1 J_0(\alpha_n r_1) S_{m-1,-1}(\alpha_n r_1) \\ &- (\alpha_n r_1) J_{-1}(\alpha_n r_1) S_{m,0}(\alpha_n r_1)) - ((m-1)\alpha_n J_0(\alpha_n) S_{m-1,-1}(\alpha_n) \\ &- (\alpha_n) J_{-1}(\alpha_n) S_{m,0}(\alpha_n))\} - J_0(\alpha_n r_1)\{((m-1)\alpha_n r_1 Y_0(\alpha_n r_1) \\ &\times S_{m-1,-1}\{\alpha_n r_1) - (\alpha_n r_1 Y_{-1}(\alpha_n r_1) S_{m,0}(\alpha_n r_1)) - ((m-1)\alpha_n Y_0 \\ &\times (\alpha_n) S_{m-1,-1}(\alpha_n) - (\alpha_n) Y_{-1}(\alpha_n) S_{m,0}(\alpha_n))\}] \qquad \ldots(2.12)\end{aligned}$$

where $S_{m,n}$ is Lommel's function (Watson 1944).

Hence

$$w = \Sigma \frac{2\alpha_n^2 J_0^2(\alpha_n r_1)\left[-\frac{\cos\alpha_n^2 t}{(\alpha_n^4 - p^2)} - \frac{ip\sin\alpha_n^2 t}{\alpha_n^2(\alpha_n^4 - p^2)} + \frac{e^{ipt}}{(\alpha_n^4 - p^2)}\right]\bar{q}}{J_0^2(\alpha_n) - J_0^2(\alpha_n r_1)}$$
$$\times [J_0(\alpha_n r)\, Y_0(\alpha_n r_1) - J_0(\alpha_n r_1)\, Y_0(\alpha_n r)]$$

where $\bar{q}(\alpha_n)$ is given by (2.12).

If $p^2 = \alpha_n^4$, the solution of equation (2.8) is

$$\bar{w} = \frac{i\bar{q}\sin\alpha_n^2 t}{2p\alpha_n^2} - \frac{i\bar{q}\, t\, e^{ipt}}{2p}.$$

Inverse Hankel Transform of $\bar{w}$ is given by

$$w = \Sigma \frac{2\alpha_n^2 J_0^2(\alpha_n r_1)\left[\dfrac{i\bar{q}\sin\alpha_n^2 t}{2p\alpha_n^2} - \dfrac{i\bar{q}te^{ipt}}{2p}\right]}{J_0^2(\alpha_n) - J_0^2(\alpha_n r_1)}$$

$$\times [J_0(\alpha_n r_1) Y_0(\alpha_n r_1) - J_0(\alpha_n r_1) Y_0(\alpha_n r)]. \quad \ldots(2.13)$$

The value of $\bar{q}(\alpha_n)$ is given by (2.12).

3. Numerical Values and Discussion

The values of w are computed in the two cases in which $p^2 \neq \alpha_n^4$ and $p^2 = \alpha_n^4$, for different values of r, t, p and $\bar{q}$. In the first case of $p^2 \neq \alpha_n^4$, when $p = 1$ the displacement profiles are sinusoidal with $|w|$ maximum at the mean radius for small values of t. But as t is increased gradually, profiles tend to become unimodal and for large t fluctuations are noted near the inner edge. $|w|$ is seen to increase with increase in the values of $\bar{q}$, i.e., as m takes values 1, 3, 5. But when $p = \pi$, the points of true maximum shift from $r = 1.4$ to $r = 1.5$ with increase in t but for comparatively larger values of t, larger displacements near the outer edge and smaller near the inner edge are noticed first and then the trend is reversed. Magnitude of w is large for large values of $\bar{q}$ in this case also.

In the second case of $p^2 = \alpha_n^4$, displacement is always positive attaining the maximum in the neighbourhood of the mean radius and displacement profiles do not vary appreciably in form with increase in t.

In case the diaphragm is isotropic, $D_1 = D_2 = H$, and then though displacement profiles can be seen to be similar in main characteristics, the numerical values of displacement differ from the anisotropic case significantly.

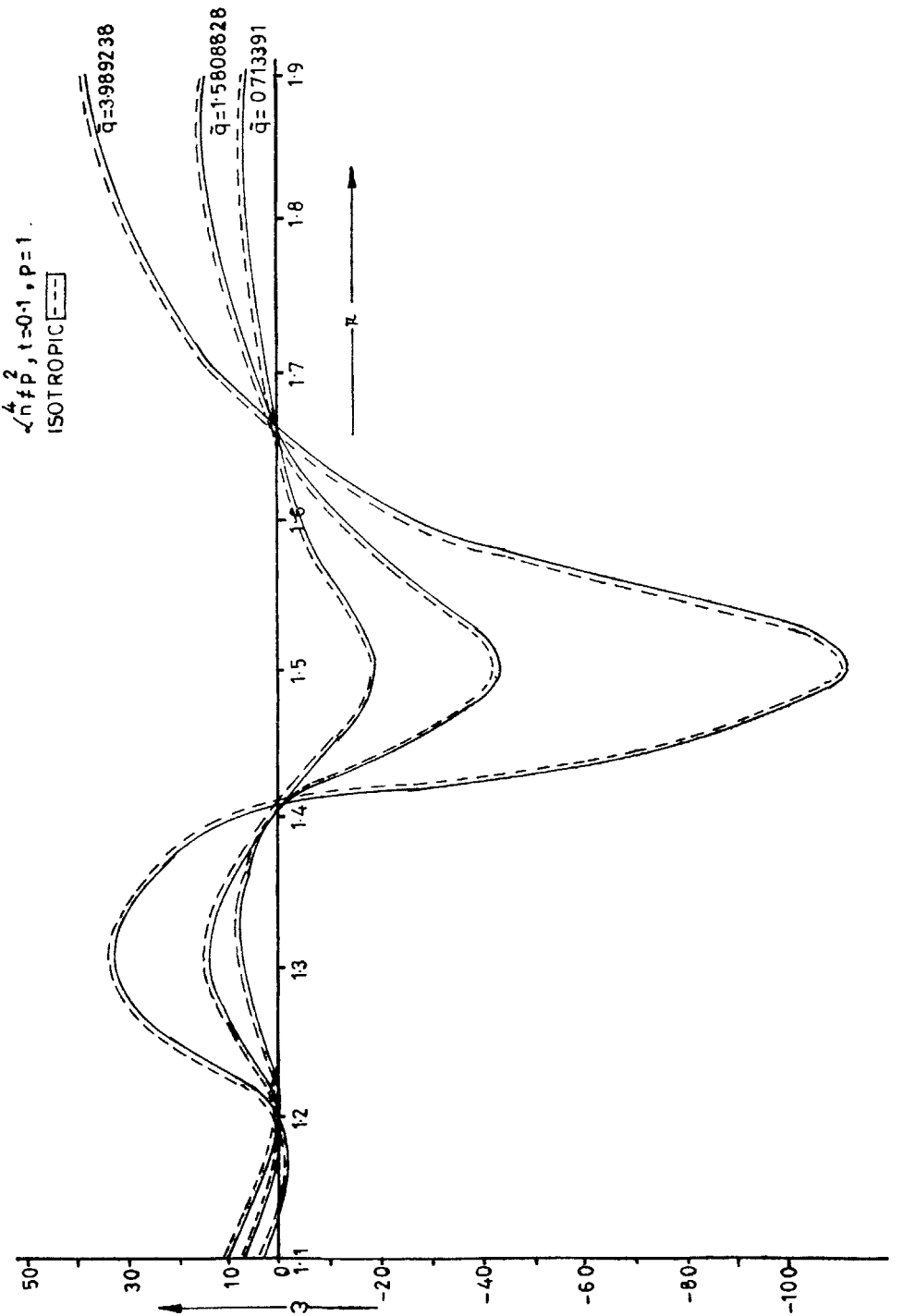
$\langle n^4 \rangle \neq \bar{p}^2$, t=0·1, p=1
ISOTROPIC [---]
q̄=3·989238
q̄=1·5808828
q̄= 0·713391
π
50
30
10
0
-20
-40
-60
-80
-100
1·1
1·2
1·3
1·4
1·5
1·6
1·7
1·8
1·9

FIG. 1

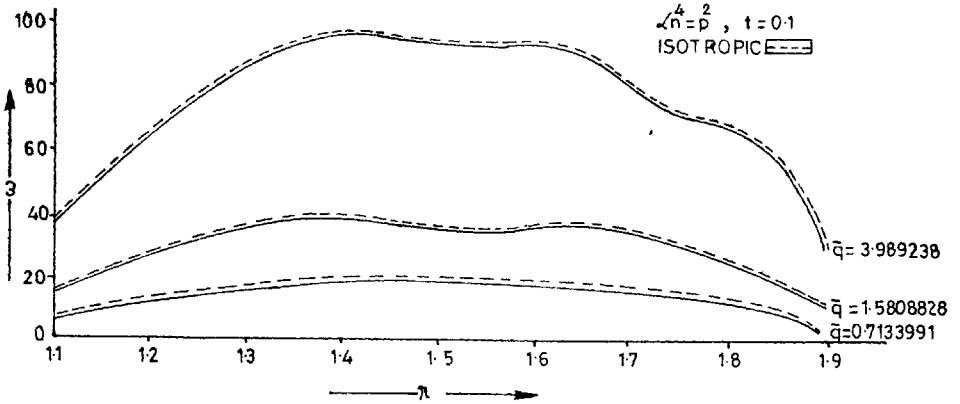
$\lambda^4 = p^2$, t = 0·1
ISOTROPIC
$\bar{q}$ = 3·989238
$\bar{q}$ = 1·5808828
$\bar{q}$ = 0·7133991
ω
π
100
80
60
40
20
0
1·1
1·2
1·3
1·4
1·5
1·6
1·7
1·8
1·9

FIG. 2.

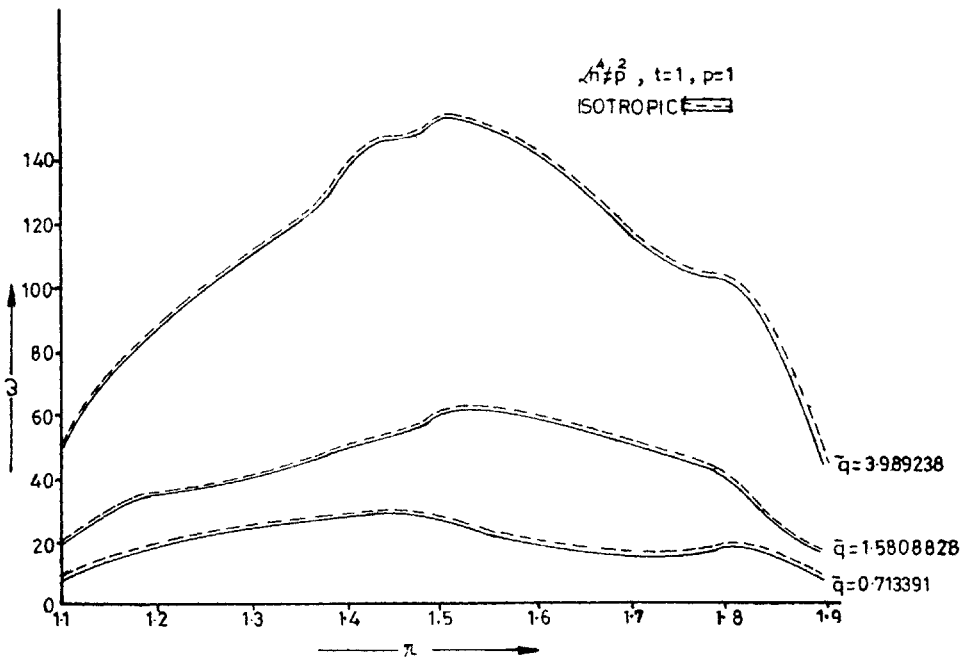
$\lambda^4 \neq p^2$, t=1 , p=1
ISOTROPIC
$\bar{q}$ = 3·989238
$\bar{q}$ = 1·5808828
$\bar{q}$ = 0·713391
ω
π
140
120
100
80
60
40
20
0
1·1
1·2
1·3
1·4
1·5
1·6
1·7
1·8
1·9

FIG. 3.

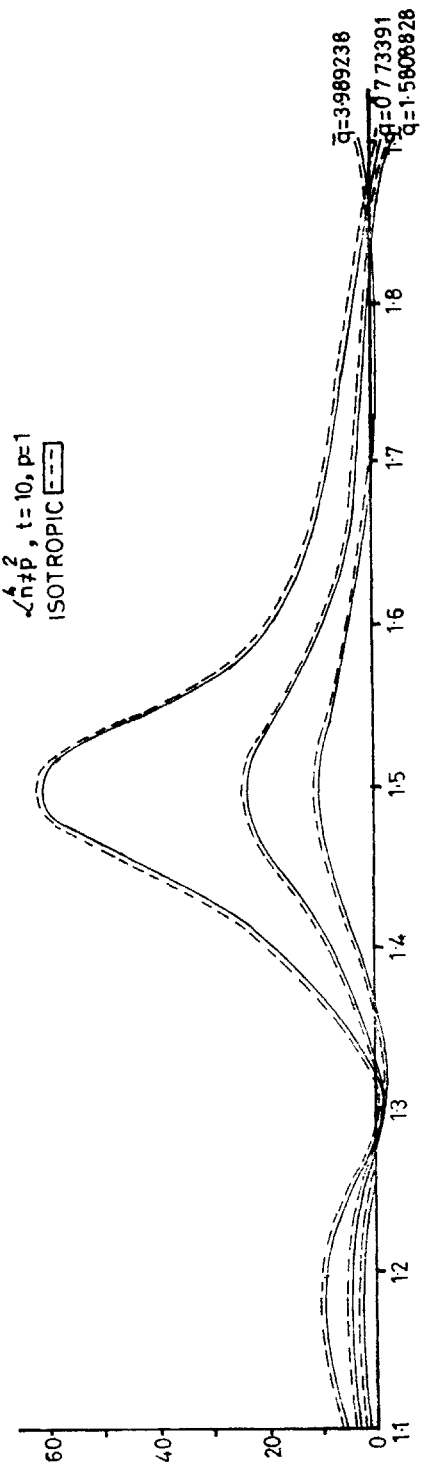
$\mathcal{L}^4_{n\neq p^2}$, t=10, p=1
ISOTROPIC
q̄=3·989238
q̄=0·773391
q̄=1·5808828
60
40
20
0
1·1
1·2
1·3
1·4
1·5
1·6
1·7
1·8

FIG. 4.

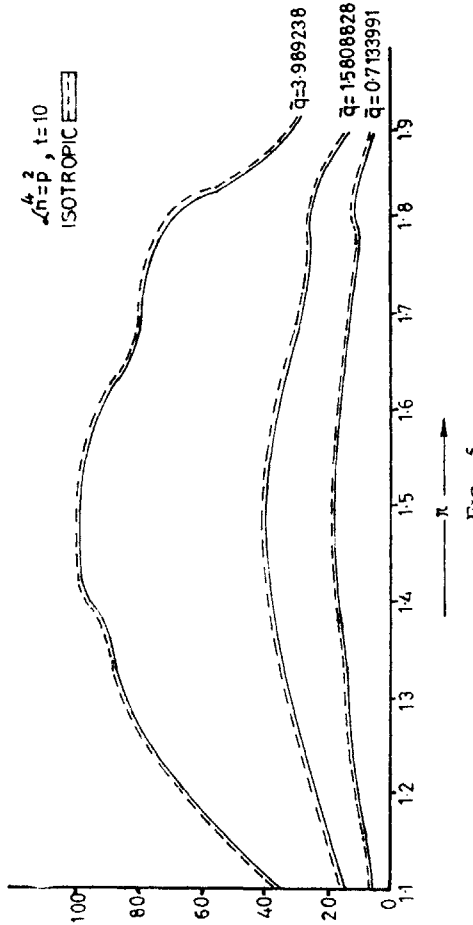
$\mathcal{L}^4_{n=p^2}$, t=10
ISOTROPIC
q̄=3·989238
q̄=1·5808828
q̄=0·7133991
100
80
60
40
20
0
1·1
1·2
1·3
1·4
1·5
1·6
1·7
1·8
1·9
π

FIG. 5.

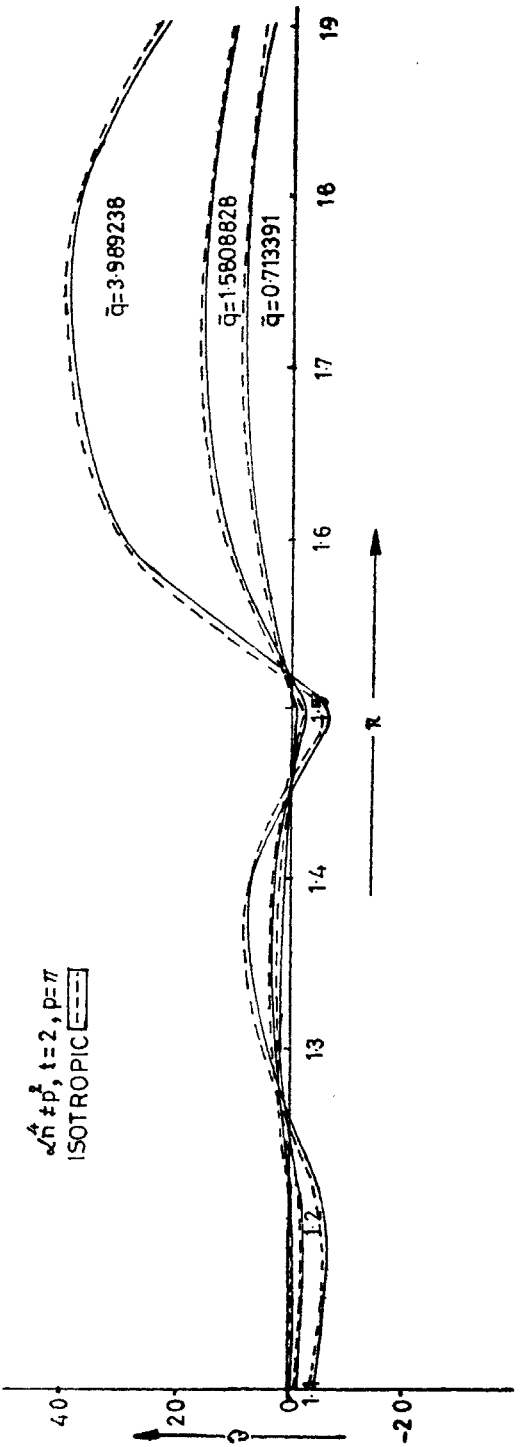
$\alpha^4 \neq p^2$, t=2, p=π
ISOTROPIC
q̄=3·989238
q̄=1·5808828
q̄=0·713391
40
20
0
-20
W
1·2
1·3
1·4
1·5
1·6
1·7
1·8
1·9
R

FIG. 6.

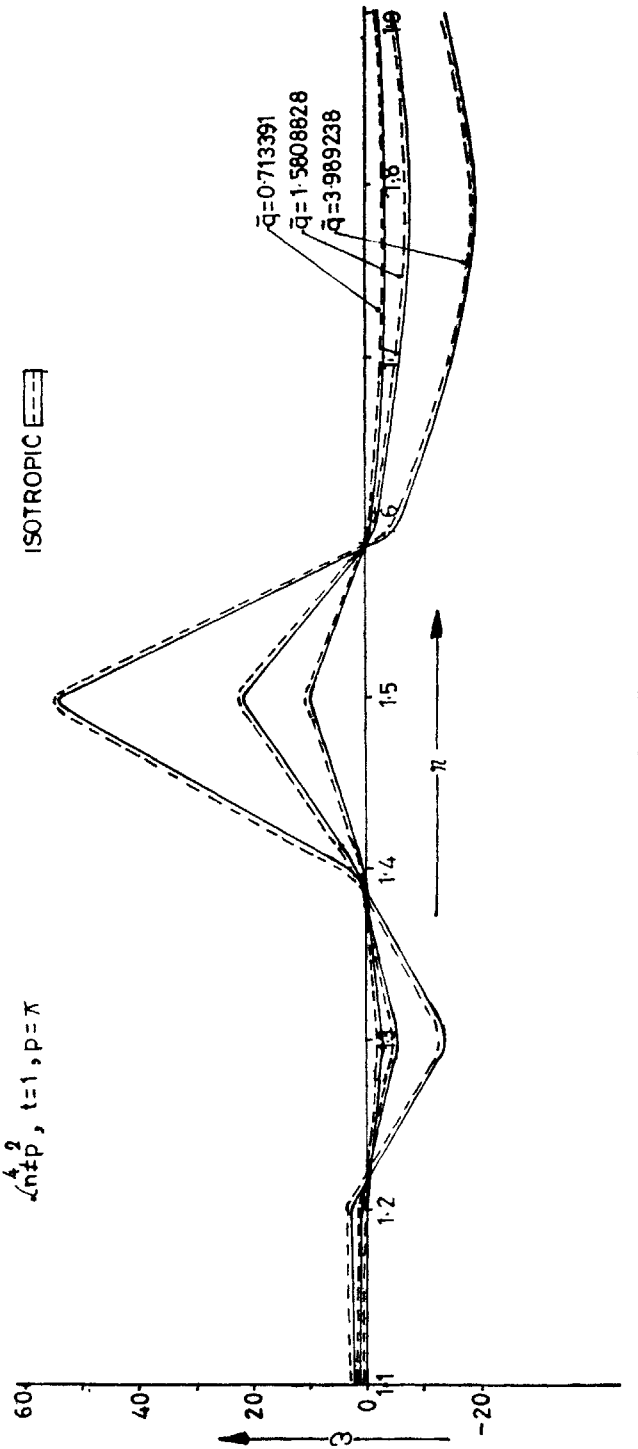
ISOTROPIC
$\bar{q}$=0·713391
$\bar{q}$=1·5808828
$\bar{q}$=3·969238
$\mathcal{L}n\bar{p}^2$, t=1, p=π
60
40
20
0
-20
1·1
1·2
1·4
1·5
1·8
η

FIG. 7.

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