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MATRIX TRANSFORMATIONS OF $l(p)$ AND bv INTO χ

S. M. SIRAJUDEEN AND M. MOHAMED ESSA

Department of Mathematics, Jamal Mohamed College, Tiruchirapalli 620020
Tamil Nadu

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Necessary and sufficient conditions have been established for an infinite matrix $A = (a_{nk})$ to transform the sequence spaces $l(p)$ and bv into $\varkappa$ where $l(p)$ is a space of Maddox (1967), bv is the well known space of sequences of bounded variation and χ is the space of Kamthan (1976).

§ 1. INTRODUCTION

If $p = (p_k)$ is a bounded sequence of strictly positive numbers, the sequence space $l(p)$ of Maddox (1967) is the generalization of the well known spaces l and l_p $(p > 1)$. Kamthan (1976) has studied the properties of the sequence space χ which we have defined in § 2.

Let (X, Y) denote the set of all matrices $A = (a_{nk})$; $n, k = 1, 2, \ldots$ which transform the sequence space X into the sequence space Y. The object of this paper is to obtain the characterization of $(l(p), \chi)$ and (bv, χ)-matrices, where bv is the space of all sequences of bounded variation. Sirajudeen (1981) has established necessary and sufficient conditions for an infinite matrix (a_{nk}) to transform the spaces l and l_p $(p > 1)$ into the space χ. These become particular cases of our theorem on $(l(p), \chi)$-matrices.

§2 deals with necessary preliminaries and some known results quoted as lemmas. Also are proved some theorems which are needed to characterize $(l(p), \chi)$ and (bv, χ)-matrices in §3.

§ 2. Writing (x_k) and Σx_k for the sequence $(x_1, x_2, \ldots, x_k, \ldots)$ and $\sum_{k=1}^{\infty} x_k$ respectively, let us list the required sequence spaces as follows.

$$l(p) = \{x = (x_k) : \Sigma \mid x_k \mid^{p_k} < \infty\}$$

$$l_\infty(p) = \{x = (x_k) : \sup_k \mid x_k \mid^{p_k} < \infty\}$$

$$bv = \{x = (x_k) : \Sigma \mid x_k - x_{k-1} \mid < \infty;\ x_0 = 0\}$$

$$\chi = \{x = (x_k) : (k! \mid x_k \mid)^{1/k} \to 0 \text{ as } k \to \infty\}.$$

When $p_k = 1$ for all k, we write $l(p)$ as l and $l_\infty(p)$ as l_∞. when $p_k = p > 1$, we write $l(p)$ as l_p.

Henceforth let N, R and C denote the set of all natural numbers, real numbers and complex numbers respectively.

Let X be a nontrivial complex linear space of elements x with zero element θ. Then $g : X \to R$ is said to be a paranorm on X if it satisfies (i) $g(\theta) = 0$; (ii) $g(-x) = g(x)$ on X; (iii) g is subadditive and (iv) for $\mu \in C$ and $x \in X$, $\mu \to \mu_0$ and $g(x - x_0) \to 0$ imply $g(\mu x - \mu_0 x_0) \to 0$ where $\mu_0 \in C$ and $x_0 \in X$.

Let $\{X_n\}$ be a sequence of subsets of X such that $\theta \in X_1$ and such that $x, y \in X_n$ imply $x \pm y \in X_{n+1}$ for $n \in N$, then (X_n) is called an α-sequence in X. If we can write $X = \bigcup_{n=1}^{\infty} X_n$ where (X_n) is a α-sequence in X and each X_n is nowhere dense in X, then X is called a α-space; otherwise X is called a β-space. Then any complete paranormed space is a β-space.

Let X^r denote the set of all continuous linear functionals on X and X^+, the generalized Köthe-Toeplitz dual of X so that

$$X^+ = \{(a_k) : \Sigma a_k x_k \text{ is convergent for every } (x_k) \in X\}$$

A linear functional A on X is an element of X^r if and only if.

$$\|A\|_M = \sup \{ | A(x) | : g(x) \leqslant 1/M\} < \infty \text{ for some } M > 1.$$

Now let us quote some known results as the following.

Lemma A (Maddox 1967 and 1969)—$l(p)$ is a linear space.

Lemma B (Maddox 1968 and 1969)—If $p = (p_k)$ is bounded with $H = \max(1, \sup_k p_k)$, $g(x) = (\Sigma | x_k |^{p_k})^{1/H}$ defines a paranorm on $l(p)$ and $l(p)$ is complete under g and (e_k) is a basis in $l(p)$ where $e_k = (0, 0, \ldots, 0, 1, 0, \ldots)$ the 1 in the kth place.

Lemma C (Maddox 1969 and Simons 1965)—(i) If $1 < p_k \leqslant H < \infty$ where $H = \sup_k p_k$ and $p_k^{-1} + s_k^{-1} = 1$ for each $k \in N$, then

$$l^+(p) = \{(a_k) : \Sigma | a_k |^{s_k} M^{-s_k} < \infty \text{ for some } M > 1\}$$

(ii) If $0 < p_k \leqslant 1$ for all k, then $l^+(p) = l_\infty(p)$.

Throughout this paper let the A transform $y = (y_n) = Ax = (A_n(x))$ of $x = (x_k)$ by the matrix $A = (a_{nk})$; $n, k = 1, 2, \ldots$ be defined by

$$y_n = A_n(x) = \Sigma a_{nk} x_k; \quad n = 1, 2, \ldots$$

Lemma D (Lascarides and Maddox 1970) — (i) Let $1 < p_k \leqslant H < \infty$ and $p_k^{-1} + s_k^{-1} = 1$ for every k. Then $A \in (l(p), l_\infty)$ if and only if there exists an integer $B > 1$ such that $\sup_n \Sigma | a_{nk} |^{s_k} . B^{-s_k} < \infty$.

(ii) Let $0 < p_k \leqslant 1$ for every k. Then $A \in (l(p), l_\infty)$ if and only if

$$\sup_{n,k} | a_{nk} |^{p_k} < \infty.$$

Lemma E (Maddox 1969 and Simons 1965) — If either $1 < p_k \leqslant H$ for all k or $0 < p_k \leqslant 1$ for all k where $H = \sup_k p_k$, then $A \in l^r(p)$ may be written as $A(x) = \Sigma \alpha_k x_k$ on $l(p)$ for some $(\alpha_k) \in l^+(p)$ and conversely $A(x) = \Sigma \alpha_k x_k$ defines an element of $l^r(p)$ for each $(\alpha_k) \in l^+(p)$.

Now we prove the following.

Theorem 1—(i) If X is a paranormed space and (A_n) is a sequence of elements of X^r, then

$$\lim_M \limsup_n (n! \, \|A_n\|_M)^{1/n} = 0 \qquad \ldots(1)$$

imply

$$(A_n(x)) \in \chi \text{ for every } x \in X. \qquad \ldots(2)$$

(ii) If X is a β-space, then (2) implies (1).

PROOF : (i) Let (1) holds. Choose any $x \in X$ and $\epsilon > 0$. There exists a $M > 1$ and a n_0 such that $(n! \, \|A_n\|_M)^{1/n} < \epsilon/2$ for $n \geqslant n_0$. Since scalar multiplication is continuous on X, there is a $K \geqslant 1$ such that $g(K^{-1}x) \leqslant 1/M$. Then we can choose $n_1 \geqslant n_0$ such that $K^{1/n} \leqslant 2$ if $n \geqslant n_1$. Hence if $n \geqslant n_1$,

$$(n! \mid A_n(x) \mid)^{1/n} = K^{1/n} (n! \mid A_n(K^{-1}x) \mid)^{1/n} < \epsilon$$

so that (2) is true.

(ii) Let X be a β-space. Choose any $\epsilon > 0$ and define for each $p \in N$,

$$X_p = \{x : x \in X \text{ and } (n! \mid 2^{-p} A_n(x) \mid)^{1/n} < \epsilon/2 \text{ for all } n \geqslant p\}.$$

Clearly θ (the zero element) $\in X_1$ and if for some $p \in N$ we have $x, y \in X_p$, then for $n \geqslant p + 1$,

$$\begin{aligned}(n! \mid 2^{-(p+1)} A_n(x \pm y) \mid)^{1/n} &\leqslant (n! \mid 2^{-(p+1)} A_n(x) \mid + n! \mid 2^{-(p+1)} A_n(y) \mid)^{1/n} \\ &\leqslant (2 \max (n! \mid 2^{-(p+1)} A_n(x) \mid, \\ &\qquad n! \mid 2^{-(p+1)} A_n(y) \mid))^{1/n} \\ &\leqslant \max ((n! \mid 2^{-p} A_n(x) \mid)^{1/n}, (n! \mid 2^{-p} A_n(y) \mid)^{1/n} \\ &< \epsilon/2.\end{aligned}$$

Thus (X_p) is a α-sequence in X. Also $X = \bigcup_{p=1}^{\infty} X_p$ and $X_p = \overline{X}_p$ for all $p \in N$ where $\overline{X}_p$ is the closure of X_p. Hence since X is a β-space, some X_B contains a sphere $S(a, \delta)$. Then if $g(x) \leqslant \delta$, we deduce $(n! \mid 2^{-B} A_n(x) \mid)^{1/n} < \epsilon$ for $n \geqslant B$. Writing $\rho = 2^{-B}$ and choosing $M > \rho^{-1}$, then by the subadditivity of g, we have $g(2^B x) < \delta$ if $g(x) < \rho$. Hence if $g(x) \leqslant 1/M$, we have

$$(n! \mid A_n(x) \mid)^{1/n} = (n! \mid 2^{-B} A_n(2^B x) \mid)^{1/n}$$

and since $\epsilon > 0$ is arbitrary, we obtain

$$\lim_M \limsup_n (n! \|A_n\|_M)^{1/n} = 0.$$

Theorem 2—$g(x) = |x_1| + \sum_{k=2}^{\infty} |x_k - x_{k-1}|$ defines a paranorm g on bv and bv is complete under g.

PROOF: Let $g(x) = |x_1| + \sum_{k=2}^{\infty} |x_k - x_{k-1}|$. If $\theta = (0, 0, \ldots)$ is the zero element in bv, then $g(\theta) = 0$, $g(-x) = g(x)$ and $g(x+y) \leqslant g(x) + g(y)$ for $x = (x_k), y = (y_k) \in bv$.

If $\mu \in C$, $x \in bv$, $\mu \to \mu_0$ and $g(x - x_0) \to 0$ for $\mu_0 \in C$ and $x_0 \in bv$, then given $\epsilon > 0$, there exists a M such that $g(x) \leqslant M$, $|\mu - \mu_0| < \epsilon/2M$ and $g(x - x_0) < \epsilon/2|\mu_0|$. Hence

$$\begin{aligned} g(\mu x - \mu_0 x_0) &\leqslant g((\mu - \mu_0)x) + g(\mu_0(x - x_0)) \\ &\leqslant |\mu - \mu_0| \, g(x) + |\mu_0| \, g(x - x_0) \\ &< \epsilon. \end{aligned}$$

Hence $g(\mu x - \mu_0 x_0) \to 0$ so that g is a paranorm on bv.

We can easily verify that bv is a complete linear metric space under the paranorm g.

Theorem 3—$bv^+ = \{(a_k) : \Sigma a_k \text{ is convergent}\}$.

PROOF: Let $S = \{(a_k) : \Sigma a_k \text{ is convergent}\}$. To prove $bv^+ \subset S$.

Let $(y_k) \in bv^+$. Take $x_k = 1$ for all k so that $(x_k) \in bv$. Then $\Sigma y_k x_k$ is convergent gives Σy_k is convergent. This implies $(y_k) \in S$ so that

$$bv^+ \subset S. \qquad \ldots(3)$$

To prove $S \subset bv^+$.

Let $(a_k) \in S$ and $(x_k) \in bv$ so that

$$\Sigma a_k \text{ is convergent} \qquad \ldots(4)$$

and

$$\Sigma |x_k - x_{k-1}| < \infty; \; x_0 = 0. \qquad \ldots(5)$$

Put $S_k = x_k - x_{k-1}$ for $k \geqslant 2$ and $S_1 = x_1$. Then

$$x_k = S_1 + S_2 + \ldots + S_k \text{ and } \Sigma |S_k| < \infty. \qquad \ldots(6)$$

Hence if $n > 1$, we have

$$\sum_{k=1}^{n} a_k x_k = \sum_{j=1}^{n} \left(S_j \sum_{i=j}^{n} a_i\right). \qquad \ldots(7)$$

Let $d_n = \sum_{j=1}^{n} \left(S_j \sum_{i=j}^{n} a_i\right)$. Then if $n > m$,

$$d_n - d_m = \sum_{j=1}^{m} S_j \Big(\sum_{i=m+1}^{n} a_i \Big) + \sum_{j=m+1}^{n} S_j \Big(\sum_{i=j}^{n} a_i \Big)$$

so that

$$| d_n - d_m | \leqslant | \sum_{m+1}^{n} a_i | \sum_{1}^{m} | S_j | + \Big(\sum_{m+1}^{n} | S_j | \Big) \Big(\sup_{m+1 < j < n} | \sum_{j}^{n} a_i | \Big).$$

Now by virtue of (4) and (6), we see that (d_n) is Cauchy and hence is convergent.

Therefore from (7), it follows that $\Sigma a_k x_k$ is convergent so that $(a_k) \in bv^+$. Hence

$$S \subset bv^+. \qquad \ldots(8)$$

Now by virtue of (3) and (8), we have $bv^+ = S$.

Theorem 4—$F \in bv^r$ may be written as

$$F(x) = \Sigma a_k (x_k - x_{k-1})$$

on bv if and only if $(a_k) \in l_\infty$ where $x_0 = 0$.

PROOF: Let $x = (x_k) \in bv$ and $F \in bv^r$ be defined by

$$F(x) = \Sigma a_k (x_k - x_{k-1}); \; x_0 = 0.$$

This implies that $\Sigma a_k (x_k - x_{k-1})$ is convergent for all $x \in bv$. To deduce $(a_k) \in l_\infty$, it suffices to show that if $\Sigma | y_k | < \infty$, then $\Sigma a_k y_k$ is convergent. However if $\Sigma | y_k | < \infty$, there is $x \in bv$ so that $y_k = x_k - x_{k-1}$; $k \geqslant 2$ and $x_1 = y_1$. Then $\Sigma a_k (x_k - x_{k-1})$ converges implies $\Sigma a_k y_k$ converges. Hence $(a_k) \in l_\infty$.

Conversely let $(a_k) \in l_\infty^{\mathbb{F}}$ and $x = (x_k) \in bv$. Then $\Sigma | x_k - x_{k-1} | < \infty$. Hence $\Sigma a_k (x_k - x_{k-1})$ is convergent converging to $F(x)$ say.

Clearly $F(x)$ is a bounded linear functional on bv. Hence $F \in bv^r$,

§ 3. MAIN RESULTS

Theorem 5—(i) When $0 < p_k \leqslant 1$, $A \in (l(p), \chi)$, if and only if

$$\lim_M \limsup_n (n \, ! \sup_k | a_{nk} | M^{-1/p_k})^{1/n} = 0 \qquad \ldots(9)$$

for some $M > 1$, and

(ii) When $1 < p_k \leqslant H < \infty$, where $H = \max (1, \sup_k p_k)$, $A \in (l(p), \chi)$ if and only if

$$\lim_M \limsup_n (n \, ! \, \Sigma | a_{nk} | {}^{q_k}D^{n \cdot q_k} \cdot B^{-q_k})^{1/n} = 0 \qquad \ldots(10)$$

for every $D \geqslant 1$ where $p_k^{-1} + q_k^{-1} = 1$ for each $k \in N$.

PROOF: By Lemma A and Lemma B, $l(p)$ is a β-space.

Case (i): Let $0 < p_k \leqslant 1$.

Necessity.

Let $A \in (l(p), \chi)$. Then for each n, $(a_{nk}) \in l^+(p) = l_\infty(p)$, by Lemma C (ii). Also by Lemma E, $A_n \in l^r(p)$ for each $n \in N$. Since $(a_{nk}) \in l_\infty(p)$, there is an $M_n > 1$ such that $|a_{nk}|^{p_k} \leqslant M_n$ for all k. Choose $M \geqslant M_n$. We have if $g(x) = \Sigma |x_k|^{p_k} < 1/M$, since $M^{1/p_k} |x_k| \leqslant 1$ for all k and since $\sup_k p_k \leqslant 1$,

$$\begin{aligned} |A_n(x)| &\leqslant \Sigma |a_{nk}| M^{-1/p_k} M^{1/p_k} |x_k| \\ &\leqslant \Sigma |a_{nk}| M^{-1/p_k} M |x_k|^{p_k} \\ &\leqslant M.\, g(x).\, \sup_k |a_{nk}| M^{-1/p_k}. \end{aligned}$$

This implies that $\|A_n\|_M \leqslant \sup_k |a_{nk}| M^{-1/p_k}$.

Now given $\epsilon > 0$, we can choose an m such that

$$|a_{nm}| M^{-1/p_k} > \sup_k |a_{nk}| M^{-1/p_k - \epsilon}.$$

Defining

$$x_k = \begin{cases} M^{-1/p_k} \operatorname{sgn}(a_{nk}); \text{ for } k = m \\ 0 \text{ ; for } k \neq m \end{cases}$$

we see that $x = (x_k) \in l(p)$, $g(x) \leqslant 1/M$ and $A_n(x) > \sup_k |a_{nk}| M^{-1/p_k} - \epsilon$

so that $\|A_n\|_M = \sup_k |a_{nk}| M^{-1/p_k}$.

Now $l(p)$ is a β-space. Hence by Theorem 1 (ii), we get (9).

Sufficiency.

Let (9) holds. Then as above we can show that for each n, $A_n \in l^\nu(p)$ with $\|A_n\|_M = \sup_k |a_{nk}| M^{-1/p_k}$ for all M such that $\|A_n\|_M$ is defined.

Hence by Theorem 1 (i), $A \in (l(p), \chi)$.

Case (ii): Let $1 < p_k \leqslant H < \infty$ where $H = \max(1, \sup_k p_k)$.

Necessity.

Let $A \in (l(p), \chi)$. Then $(a_{nk}) \in l^+(p)$ and hence by Lemma E, $A_n \in l^r(p)$ for all n. If $A \in (l(p), \chi)$, the $(D^n a_{nk}) \in (l(p), \chi)$ for all $D > 1$. Hence it is enough to show that the condition holds for $D = 1$. Now $\chi \subset l_\infty$. Hence using Lemma D, there is a $B > 1$ such that

$$T_n = \Sigma |a_{nk}|^{q_k} B^{-q_k} \leqslant 1 \text{ for every } n \in N.$$

Choose any n and define

$$x_k^{(n)} = B^{-q_k} \mid a_{nk} \mid^{q_k^{-1}} \operatorname{sgn}(a_{nk}) \text{ for each } k, \text{ then}$$

$$g^H(x^{(n)}) = \Sigma B^{-q_k - p_k} \mid a_{nk} \mid^{q_k} \leqslant B^{-H} T_n \leqslant B^{-H}$$

and

$$A_n(x^{(n)}) = T_n \text{ whence } \|A_n\|_B \geqslant T_n \text{ for each } n.$$

Hence by Theorem 1 (ii), since $l(p)$ is a β-space, we get (10) for $D = 1$.

Sufficiency.

Let (10) holds. Then choose $\epsilon > 0$ and $D < 2/\epsilon$. There exists a $B > 1$ and n_1 such that for $n \geqslant n_1$,

$$(n! \Sigma \mid a_{nk} \mid^{q_k} D^{n \cdot q_k} B^{-q_k})^{1/n} < \frac{\epsilon}{2}.$$

We know that if $r > 1$ and $r^{-1} + s^{-1} = 1$, then $\mid xy \mid \leqslant \mid x \mid^r + \mid y \mid^s$. Hence if $g(x) \leqslant 1/B$ and $n \geqslant n_1$, we have

$$\begin{aligned}
(n! \mid A_n(x) \mid)^{1/n} &\leqslant (n! \Sigma \mid a_{nk} \mid D^n B^{-1} B D^{-n} \mid x_k \mid)^{1/n} \\
&\leqslant \Sigma (n! \mid a_{nk} \mid^{q_k} D^{n \cdot q_k} B^{-q_k + n!} B^{p_k} D^{-n \cdot p_k} \mid x \mid^{p_k})^{1/n} \\
&< \frac{\epsilon}{2} + (n! D^{-n} B^H g^H(x))^{1/n} \\
&< \frac{\epsilon}{2} + (n! D^{-n})^{1/n} \\
&< \epsilon.
\end{aligned}$$

Hence $A \in (l(p), \chi)$.

Corollary 1 (Sirajudeen 1981)—$A \in (l, \chi)$ if and only if

$(n! \mid a_{nk} \mid)^{1/n} \to 0$ uniformly in k as $n \to \infty$.

PROOF: Take $p_k = 1$ for all k.

Corollary 2 (Sirajudeen 1981) — When $p > 1$, $A \in (l_p, \chi)$ if and only if $(n! \Sigma \mid a_{nk} \mid^q)^{1/n} \to 0$ as $n \to \infty$ where $p^{-1} + q^{-1} = 1$.

PROOF: Take $p_k = p > 1$ for all k so that $q_k = q$.

Theorem 6—$A \in (bv, \chi)$ if and only if

$$(n! \mid \sum_{k=j}^{\infty} a_{nk} \mid)^{1/n} \to 0 \text{ uniformly in } j \text{ as } n \to \infty \qquad \ldots(11)$$

PROOF: Sufficiency.

Let (11) holds and $x = (x_k) \in bv$. Defining $b_{nj} = \sum_{k=j}^{\infty} a_{nk}$ so that $b_{nj} \to 0$ as $\to \infty$. Then $(b_{nj})_{j=1}^{\infty}$ is well defined. From (11), we have

$$(n! \mid b_{nj} \mid)^{1/n} \to 0 \text{ uniformly in j as } n \to \infty \qquad \ldots(12)$$

and $(b_{nj})_j \in l_\infty$ for each n.

Now

$$A_n(x) = \Sigma a_{nk}x_k = \operatorname*{iim}_{R\to\infty} \sum_{k=1}^{R} a_{nk}x_k$$

$$= \lim_{R\to\infty} \left[\sum_{j=1}^{\infty} b_{nj}(x_j - x_{j-1}) + b_{n,R+1}x_R\right]$$

$$= \sum_{j=1}^{\infty} b_{nj}(x_j - x_{j-1}) \text{ [Since } b_{nR} \to 0 \text{ as } R \to \infty \text{ and } (x_k) \in bv \subset l_\infty]$$

Now $(b_{nj})_j \in l_\infty$ for each n. Hence by Theorem 4, $A_n \in bv^r$. Also there is an $M_n > 1$ such that $|b_{nj}| \leqslant M_n$ for all j. Now by Theorem 2, bv is a paranormed space paranormed by g given by $g(x) = |x_1| + \sum_{j=2}^{\infty} |x_j - x_{j-1}|$. Choose $M \geqslant M_n$. We have if $g(x) \leqslant 1/M$, then

$$|A_n(x)| \leqslant \sup_j |b_{nj}| \left(|x_1| + \sum_{j=2}^{\infty} |x_j - x_{j-1}|\right)$$

$$\leqslant \sup_j |b_{nj}| M^{-1}.$$

Hence

$$\|A_n\|_M \leqslant \sup_j |b_{nj}| M^{-1}. \qquad \ldots(13)$$

Now given $\epsilon > 0$, we can choose an m such that

$$|b_{nm}| M^{-1} > \sup_j |b_{nj}| M^{-1} - \epsilon.$$

Defining

$$x_j = \begin{cases} 0 \ ; (j < m) \\ M^{-1} \operatorname{sgn}(b_{nj}) \ ; (j \geqslant m). \end{cases}$$

We have $(x_j) \in bv$ and $g(x) \leqslant M^{-1}$. Then

$$A_n(x) = |b_{nm}| M^{-1} > \sup_j |b_{nj}| M^{-1} - \epsilon.$$

Hence from this and (13), we have

$$\|A_n\|_M = \sup_j |b_{nj}| M^{-1}. \qquad \ldots(14)$$

Then from (14), (12) and Theorem 1 (i), we get $A \in (bv, \chi)$.

Necessity.

Let $A \in (bv, \chi)$. Then $(a_{nk}) \in bv^+$ for each n so that by Theorem 3, Σa_{nk} is convergent. Hence defining $b_{nj} = \sum_{k=j}^{\infty} a_{nk}$, as in the sufficiency part, for each n, we have $A_n \in bv^r$ defined by $A_n(x) = \sum_{j=1}^{\infty} b_{nj}(x_j - x_{j-1})$ and

$$\|A_n\|_M = \sup_j | b_{nj} | M^{-1} = \sup_j | \sum_{k=j}^{\infty} a_{nk} | M^{-1}$$

for all M such that $\|A_n\|_M$ is defined.

Now by Theorem 2, bv is a complete paranormed space. Hence bv is a β-space so that by Theorem 1 (ii), we have (11).

References

Kamthan, P. K. (1976). Bases in a certain class of Frechet space. *Tamkang J. M.*, 41–49.

Lascarides, C. G., and Maddox, I. J. (1970). Matrix transformations between some classes of sequences. *Proc. Camb. Phil. Soc.*, 99–104.

Maddox, I. J. (1967). Spaces of strongly summable sequences. *Quarterly J. Math. Oxford*, **68**, 345–55.

——— (1968). Paranormed sequence spaces generated by infinite matrices. *Proc. Camb. Phil. Soc.*, **64**, 335–40.

——— (1969). Continuous and Köthe-Toeplitz duals of certain sequence spaces. *Proc. Camb. Phil. Soc.*, **65**, 431–35.

——— (1969). Some properties of paranormed sequence spaces. *J. London. Math. Soc.*, **1**, 316–22.

Simons, S. (1965). The sequence spaces $l(p_\nu)$ and $m(p_\nu)$. *Proc. London. Math. Soc.*, **15**, 422–36.

Sirajudeen, S. M. (1981). Matrix transformations of $c_0(p)$, $l^\infty(p)$, l^p and l into χ. *Indian J. pure appl. Math.* **12**, 1106–13.