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ORDERED TOPOLOGICAL SPACES AND THE THEOREMS OF DEBREU AND PELEG

GHANSHYAM MEHTA

Department of Economics, University of Queensland, St. Lucia, Queensland, 4067, Australia

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The object of this paper is to investigate the relationship between order-denseness and topology for completely preordered topological spaces and for partially ordered topological spaces. After obtaining some results in this area, applications are made in the field of utility theory. It is proved that a completely preordered second countable topological space can be embedded in an order-separable space and generalizations of the Debreu theorems are obtained.

INTRODUCTION

This paper investigates the relationship between order-denseness and topology for completely preordered topological spaces and for strictly partially ordered topological spaces. After obtaining some results in this area applications are made in the field of utility theory.

In Section 1, some preliminary concepts are defined. In Section 2, the relation between order-denseness and topology is studied. In Section 3, it is proved that a completely preordered second countable space can be embedded in an order-separable space. This enables us to obtain another generalization of the classic theorems of Debreu and Peleg in utility theory.

PRELIMINARIES

Let X be a set and suppose $\lesssim$ is a binary relation on X. $\lesssim$ is said to be a preference relation on X if it satisfies the following conditions :

(a) $\lesssim$ is reflexive

(b) $\lesssim$ is transitive

(c) $\lesssim$ is complete, i. e. for every $x, y \in X$, either $x \lesssim y$ or $y \lesssim x$. A binary relation satisfying (a) and (b) is said to be preordering on X. A preference relation $\lesssim$ on

X is consequently, a complete preordering on X. $(X, \precsim)$ is said to be a completely preordered or linearly preordered set.

A utility function for $(X, \precsim)$ is a real-valued function such that $f(x) \leqq f(y)$ if and only if $x \precsim y$. A utility function is an order-preserving real function on a completely preordered set. Utility functions on partially ordered spaces will be defined later.

A preordering set $\precsim$ on a set X is an 'ordering' if it satisfies the following condition:

$$x \precsim y,\ y \precsim x \Rightarrow x = y \ (\precsim \text{ is antisymmetric}).$$

A complete preordering on X has associated with it two other binary relations.

Define '$x \sim y$' as $x \precsim y$ and $y \precsim x$. Then it is easy to see that $\sim$ is an equivalence relation. Then $X/\sim$ is the set of equivalence classes of $\sim$. These equivalence classes are called indifference classes.

If f is a utility function, then $x \sim y$, iff $f(x) = f(y)$.

Define '$x < y$' as $x \precsim y$ and not $y \precsim x$. $y > x$ means the same thing as $x < y$.

$<$ is a strict preference relation on X. Clearly, $<$ is only a 'partial order' on X since it is possible to have $x \sim y$.

Define a new relation $<^1$ on $X/\sim$ by $x <^1 y$ iff $a < b \forall a \in X,\ b \in Y$. Then $<^1$ is a linear ordering on $X/\sim$.

Suppose now that $(X, \precsim)$ is a completely preordered topological space. Let f be a continuous utility function on X. Since for each real number α, the sets $\{x/x \geqq \alpha\}$ and $\{x/x \leqq \alpha\}$ are closed, the inverse images of these sets under f must be closed. Hence, a necessary condition for the existence of continuous utility functions on $(X, \precsim)$ is that for all $y \in X$, the sets $\{x \in X/x \succsim y\}$ and $\{x \in X/x \precsim y\}$ must be closed. A topology μ for X in which these sets are closed is called a natural topology for $(X, \precsim)$.

Let $(X, \precsim)$ be completely preordered. The open intervals of $\precsim$ are sets of the form $\{x/x < y, y \in X\}$, $\{x/x > y, y \in X\}$, $\{x/y < x < z$, and $y, z \in X\}$.

A natural topology for $(X, \precsim)$ is finer than the topology generated by the open intervals of $\precsim$.

An upper preorder topology for $(X, \precsim)$ is a topology in which sets of the form $\{x/x < y,\ y \in X\}$ are open. Similarly, a lower preorder topology for $(X, \precsim)$ is one in which sets of the form $\{x/x > y,\ y \in X\}$ are open.

Example—Let $X = R^2_+ = \{(x, y) \in R^2/x \geqslant 0,\ y \geqslant 0\}$ be the non-negative quadrant of Euclidean space R^2. Let $\leqslant$ denote the lexicographic preference relation on X, i.e. $(a, b) \precsim (a^1, b^1)$ if and only if either $a \leqslant a^1$, or if $a = a^1$ and $b \leqslant b^1$. Then $(X, \precsim)$ does not admit a utility representation. We shall assume that R^2_+ has the usual relativized topology from R^2. Let $z \in X$.

The set $\{x/z \precsim x\}$ consists of the vertical line segment above z and all points to the right. It does not include all its boundary points. Hence $\{x/z \precsim x\}$ is not closed. Similarly, $\{x/x \precsim z\}$ is not closed. This means that the usual topology for R^2 is not a

natural topology. Hence, $(X, \precsim)$ with the usual topology cannot be represented by a continuous utility function, or even a semicontinuous utility function.

We will now prove that there is no utility function on $(X, \precsim)$. Suppose there exists a utility function f. For each $x_0 \in R_+$ consider the half-line $L_{x_0} = \{(x, y) \in R^2_+ / x = x_0\}$.

Let $0 \leqslant a < x_0 < b$. Then $f(L_{x_0})$ is bounded below by $f(a, 0)$ and bounded above by $f(b, 0)$ because f is a utility function. For each $x_0 > 0$, let $m_{x_0} = \inf\{f(x,y)/(x, y) \in L_{x_0}\}$ and $M_{x_0} = \sup\{f(x, y)/(x, y) \in L_{x_0}\}$. For each positive real number x we have an interval (m_x, M_x). The family of intervals $\{(m_x, M_x)/x$, a positive real number$\}$ has the following properties:

(a) For each x, $m_x < M_x$ since L_x has infinitely many points.

(b) If $x < y$, $M_x < m_y$.

(c) It is countable since the rationals are dense in the reals.

Hence, we have a bijection between a countable set and an uncountable set, which is a contradiction.

Order-Denseness and Topology

We now investigate the relation between order-denseness and the topology of a preordered space. Initially, we shall restrict the discussion to completely preordered spaces. We shall consider later the extent to which these results continue to hold for partially ordered spaces.

Let $(X, \precsim)$ be a completely preordered set. A subset Z is said to be perfectly order-dense if for $x, y \in X$ with $x \prec y$ $\exists$ $z \in Z$ such that $x \precsim z \precsim y$. $(X, \precsim)$ is said to be perfectly (order) separable if it has a countable perfectly order dense subset.

A completely preordered set $(X, \precsim)$ is said to be weakly separable, if $\exists$ a countable subset Z such that if $x, y \in X$ and $x \prec y$ with $(x, y) \neq \phi$, then $z \in Z$ such that $x \prec z \prec y$.

A completely preordered set $(X, \precsim)$ is said to be order separable if $\exists$ a countable subset Z such that if $x, y \in X$ and $x \prec y$, then $\exists$ $z \in Z$ such that $x \prec z \prec y$.

Proposition 1—Let $(X, \precsim)$ be a completely preordered topological space with the topology generated by the open intervals of $\precsim$. Then $(X, \precsim)$ is separable topological space if and only if $(X, \precsim)$ is weakly separable.

Proof : Assume $(X, \precsim)$ is topologically separable. Let Z be a countable dense subset. Let $x \prec y$ with $(x, y) \neq \phi$. Then the open interval (x, y) is an open set. Since it is non-empty and Z is dense, $\exists$ $z \in Z$ such that $x \prec z \prec y$. This proves that it is weakly separable.

Conversely, suppose $(X, \precsim)$ is weakly separable. Let Z be a countable order-dense set. Suppose U is an open set and $x \in U$. Since sets of the form $a \in X/a < m$, $m \in X\}$ and $\{a \in X/n < a$, $n \in X\}$ are members of the sub-base and U is open, $\exists$ a

member of the base $\{a \in X/n<a<m\}$ which owns x and is entirely contained in U. Since, $(n, m) \neq \phi$, $\exists\ z \in Z$ such that $n < z < m$ by weak separability, so $z \in U$. This proves that each non-empty set contains an element from Z. Consequently, Z is dense and X is topologically separable.

Remark: The completeness of the preference relation does not seem to be an essential consideration in the above proof. This suggests that the result is also true for partially ordered spaces.

Proposition 2—Let $(X, \preccurlyeq)$ be a completely preordered topological space with the topology generated by the open intervals of $\preccurlyeq$. Then $(X, \preccurlyeq)$ is perfectly topologically separable if and only if it is perfectly order-separable.

PROOF: Suppose that $(X, \preccurlyeq)$ is second countable or perfectly topologically separable. Let $B = \{B_1, B_2, \ldots\}$ be a countable base. Select $z_i \in B_i$ to get a countable dense set Z. Let $X/\sim$ be the set of indifference classes.

Consider now pairs of elements a, b such that $a < b$ and the open interval $(a, b) = \phi$. With each such pair, a, b we associate a member of the base B_b as follows. The set $\{x \in X/x < b\}$ is open and owns a. Since B is a base, $\exists\ B_b \in B$ such that $a \in B_b \subseteq \{x \in X/x < b\}$. If a^1, b^1 is another pairs of points such that $a^1 < b^1$ and $(a^1, b^1) = \phi$ we associate $B_b{}^1 \in B$ with this pair in the manner described above. Then $B_b \neq B_b{}^1$ for if $a < b \preccurlyeq a^1 < b^1$, $a^1 \in B_b{}^1$ but $a^1 \notin B_b$ and if $a^1 < b^1 \preccurlyeq a < b$ then $a \in B_b$, $a \notin B_b{}^1$. Since the preordering is complete there are no other possibilities. Hence, we have shown that there is an injection from the set of all pairs a, b with $a < b$ and $(a, b) = \phi$ to the countable base B. This proves that the set of all such pairs is countable. Observe that essential use is made of the completeness of the preordering to get the result.*

Consequently, the collection of all pairs of points, or more precisely pairs of equivalence classes of points with the property described above, is countable. Choose an element, x^1 in class a^1 to form a countable set Z^1. Then $Z \cup Z^1$ is a countable set. To prove $(X, \preccurlyeq)$ is perfectly order separable let $x, y \in X$ such that $x < y$. If the open set (x, y) is not empty, it contains an element $z \in Z$ since Z is topologically dense. If the set $(x, y) = \phi$. $x \sim x^1$ for some $x^1 \in Z^1$ and $y \sim y^1$, $y^1 \in Z^1$. Hence, $\exists$ an element z in Z^1 such that $x \preccurlyeq z \preccurlyeq y$. So that in any case, $\exists$ an element $z \in Z \cup Z^1$ such that $x \preccurlyeq z \preccurlyeq y$. This proves that $(X, \preccurlyeq)$ is perfectly order separable.

Conversely, suppose $(X, \preccurlyeq)$ is perfectly order separable with a countable perfectly dense subset Z. Let U be an open set with $x \in U$. Since U is open, $\exists$ a member of the base, say B, such that $x \in B = \{q/a < q < b\} \subseteq U$. Since X is

* With a partial order "interlacing", may occur. For example, if a, b and a^1, b^1 are two pairs, we may have $a^1 < b$, $a < b^1$ with a not comparable with a^1.

perfectly order separable and $a<x<b$, $\exists$, $z_i, z_j \in Z$ such that $a \leqslant z_i \leqslant x \leqslant z_j \leqslant b$. Two cases may arise :

(1) $z_i < x < z_j$

(2) $z_i \tilde{} x$, $x \tilde{} z_j$ or both.

In case (2), $(a, z_i) = \phi$, or $(z_j, b) = \phi$ or both because otherwise we are back with case (1).

If $(a, z_i) = \phi$, a is the immediate predecessor of z_i. Hence, the indifference class of z_i may be associated in a one-to-one fashion with the indifference class of a. Similarly, the class of z_j is associated with the class of b.

Consider now the following two families of sets:

$$S_1 = \{x/z_i < x < z_j\} \; i, j = 1, 2, 3, \ldots$$

This is a countable family of open sets. $S_2 = \{x/f_1(z_i) < x < f_2(z_j)\}$ where $f_1(z_i)$, $i = 1, 2, 3, \ldots$ is the immediate predecessor (if there exists one) of z_i, and $f_2(z_j)$, $j = 1, 2, 3, \ldots$ is the immediate successor of z_j (if there is one). Again, S_2 is a countable family of open sets. Let $S = S_1 \cup S_2$. Then S is a countable base for the topology of $(X, \leqslant)$. Thus $(X, \leqslant)$ is second countable.

In our discussion up to now we have regarded preferences as complete preorderings. We now adopt a different approach by regarding preference as strict partial orders.

A strict partial order $<$ on a set X is a binary relation on X which is irreflexive and transitive. Consequently, a strict partial order is also asymmetric.

Define $x \tilde{} y$ if and only if not $x < y$ and not $y < x$. Thus '$\tilde{}$' means not comparable. It does not mean indifference as usually interpreted.

It is possible that $x \tilde{} y$, $y \tilde{} z$ and that $x \Psi z$. So $\tilde{}$ is not an equivalence relation. Define $x \approx y$ iff $\forall z$, $z < x \Leftrightarrow z < y$ and $x < z \Leftrightarrow y < z$. Then $\approx$ is an equivalence relation.

A utility function for $(X, <)$ is a real function such that $x < y \Rightarrow u(x) < u(y)$. Since $<$ is a partial order, the converse implication may not hold, when $x \tilde{} y$.

Let $(X, <)$ be a strictly partially ordered topological space. Define $x \overset{\approx}{<} y$ to mean either $x < y$ or $x \approx y$.*

Then $(X, <)$ is said to be perfectly order separable if there exists a countable subset Z such that if $x, y \in X$ and $x \overset{\approx}{<} y$, $\exists\, z \in Z$ such that $x \overset{\approx}{<} z \overset{\approx}{<} y$. A set such as Z is called a perfectly dense set. $(X, <)$ is said to be weakly separable if $\exists$ a countable subset Z such that if $x, y \in X$, $x < y$ and $(x, y) \neq \phi$, then $\exists\, z \in Z$ such that $x < z < y$. $(X, <)$ is said to be separable if $\exists$ a countable subset Z with the property that if $x < y$, then $\exists\, z \in Z$ such that $x < z < y$.

* Observe that if $(X, \leqslant)$ is completely preordered, $\approx$ is the same as $\sim$. So the definitions of order-denseness agree with the ones given earlier.

Let $(X, <)$ be a strictly partially ordered topological space. The order $<$ on X is said to be continuous if for all $y \in X$, the set $\{x \in X / x < y\} = L_y$ open. The partial order $<$ is said to be strongly continuous if, in addition, the set $U_y = \{x \in X / y < x\}$ is open for all $y \in X$. Denote the topological closure of a subset A of X by $\overline{A}$. Then $<$ is said to be spacious if $x < y$ implies $L_y \supseteq \overline{L}_x$.

Proposition 3—Let $(X, <)$ be a strictly partially ordered topological space. Let the topology of X be generated by sets of the form L_z and U_z. Then $(X, <)$ is a separable topological space if and only if $(X, <)$ is weakly separable.

PROOF : The proof of Proposition 1 is valid in this case if we replace $\lesssim$ by $\overset{\approx}{<}$.

Proposition 4—Let $(X, <)$ be a strictly partially ordered topological space. Let the topology of X be generated by sets of the form L_z and U_z. If $(X, <)$ is perfectly order separable, then $(X, <)$ is second countable.

PROOF : The proof of the second half of proposition 2 is still valid in this case if we replace $\lesssim$ by $\overset{\approx}{<}$.

Remark—Let $(X, \lesssim)$ be a completely preordered topological space with the topology generated by the open intervals of $\lesssim$. Then we have the following set of implications :

$$\begin{array}{ccccc} \text{Separable} & \Longrightarrow & \text{Perfectly Order Separable} & \Longrightarrow & \text{Weakly Separable} \\ 1 & & 2 & & 3 \\ & & \updownarrow & & \updownarrow \\ & & \text{Second Countable} & & \text{Topologically Separable} \\ & & 4 & & 5 \end{array}$$

(1) $\Longrightarrow$ (2) trivially.

(2) and (4) are equivalent by Proposition 2

(3) and (5) are equivalent by Proposition 1

Since a second countable topological space is separable, we also have (2) $\Rightarrow$ (3).

Examples— (a) Let $X = R_+^2$ with the lexicographic ordering. Then X is a preordered space which is not weakly separable.

(b) Let $X = \{0, 1\}$ with the usual preordering. Then X is perfectly order separable but not separable.

(c) With each irrational number ξ associate an "open gap" (a_ξ, b_ξ), $a_\xi < b_\xi$. Let X consist of all the rational numbers and the end points of the open gaps a_ξ, b_ξ, ξ an irrational number. The ordering on X is the usual ordering. If q is a rational number and ξ irrational with $q < a_\xi$, then $q < a_\xi < b_\xi$. Similarly, if $\xi < q$. If ξ, μ are two irrational numbers and $\xi < \mu$ then $a_\xi < b_\xi < a_\mu < b_\mu$. Then X is weakly separable but not perfectly order separable.

Observe, however, that if $(X, \lesssim)$ is weakly separable and connected, then it is perfectly order separable. To prove this, let $x < y$. Since $\{a \in X / a \lesssim X\}$ and

$\{a \in X/y \lesssim a\}$ are disjoint and closed they do not exhaust X. Hence the open interval $(x, y) \neq \phi$.

Suppose now that $(X, <)$ is a strictly partially ordered topological space with the interval topology. Then we have the following set of implications:

Separable $\Rightarrow$ Perfectly Order Separable $\Rightarrow$ Weakly Separable

1	2	3
	$\Downarrow$	$\updownarrow$
	Second Countable	Topologically Separable
	4	5

(1) $\Rightarrow$ (2) trivially.

(2) $\Rightarrow$ (4) by Proposition 4.

(3) $\Leftrightarrow$ (5) by Proposition 3.

Since a second countable topological space is separable, a perfectly order separable space is weakly separable.

Utility Functions

For completely preordered spaces there are two basic results proved by Debreu (1954) that show the conditions under which such spaces have continuous utility representations.

Theorem 1 Debreu (1954)—Let $(X, \lesssim)$ be a completely preordered topological space that is connected and topologically separable. Then $(X, \lesssim)$ has a continuous utility representation in any natural topology.

Theorem 2 (Debreu 1954)—Let $(X, \lesssim)$ be a completely preordered topological space satisfying the second axiom of countability. Then $(X, \lesssim)$ has a continuous utility representation in any natural topology.

For partially ordered topological spaces the following theorem was proved by Peleg (1970).

Theorem 3 (Peleg 1970)—Let $<$ be a partial order on a topological space X. If the order is open, separable and spacious then there exists a continuous utility function for $(X, <)$.

Since each complete preordering has an associated strict order part it is natural to conjecture that Debreu's theorem can be derived from Peleg's theorem. This conjecture was partly proved by Lee (1972).

The nub of Lee's argument is as follows. Let $(X, \lesssim)$ be completely preordered. Then if $X/\sim$ is non-trivial one can define a new strict partial order on $X/\sim$ by saying that $a(x)$, the indifference class of x, is strictly preferred to $a(y)$, the indifference class of y, if and only if any member of $a(x)$ is preferred to any member of (ay). We

now endow the set $X/\sim$ with the quotient topology. Suppose now that $(X, \precsim)$ satisfies the conditions of Debreu's first fundamental theorem. Since X is connected and separable the countable dense set $Z = z_1, z_2, \ldots$ has the following property. If $x < y$ then there is a z in Z such that $x < z < y$. Hence, in this case, the space $(X, \precsim)$ is not only perfectly order-separable. By the definition of the order on $X/\sim$ it follows that $X/\sim$ with the quotient topology is also order-separable. By the definition of the quotient topology on $X/\sim$ it easily follows that $<$ is strongly open, which implies that $<$ is spacious since the ordering on $X/\sim$ is complete. Hence, all the conditions of Peleg's theorem are satisfied. Therefore, there exists a continuous utility function T on the quotient space $X/\sim$. Now Lee proves that if there is a continuous utility function (in the sense of Peleg) on the quotient space $X/\sim$ then there is a continuous utility function F on $(X, \precsim)$. This function F is obtained by composing T with the canonical mapping, i.e. $F = TA$. This proves Debreu's theorem as a consequence of Peleg's theorem.

We see from the above that Lee has demonstrated that Peleg's theorem implies the existence of a continuous utility function on any order-separable space with a natural topology. Lee then shows that a connected, separable topological space is order-separable and thus deduces Debreu's Theorem 1 from Peleg's theorem. Lee left unanswered the question about the relationship of Peleg's theorem to Debreu's other theorem on second countable spaces. It would seem from Lee's note that only Debreu's Theorem 1 follows from Peleg's theorem.

We will now prove that Peleg's theorem can be regarded as a generalization of even the Debreu theorem on second countable spaces. This will constitute an extension of Lee's work.

It is not hard to find examples of completely preordered second countable topological spaces that are not order-separable. Hence, Lee's argument cannot be applied directly to X. If we can enlarge X to a space X' that is order-separable, then we can apply Lee's results to X' and conclude that X' (therefore, X) has a continuous utility representation.

Let X be a second countable space and let $Y = X/\sim$ be the set of indifference classes of X. Then Y is said to have a jump (a, b) if a and b belong to Y and there is no c in Y such that $a < c < b$. It can easily be proved that second countability implies that there are only countably many jumps (a_n, b_n). Enlarge Y by interposing between each jump (a_n, b_n), $n = 1, 2, 3, \ldots$, the open real interval $(n, n + 1)$ with its usual ordering. Call this enlarged space Y'. Corresponding to Y' is the enlarged space X'. The topology of X' is obtained in a natural way from X and the topology of the intervals $(n, n + 1)$.

Since X is second countable, it has a countable dense set Z'. Let Z'' be the set of rational numbers and define Z to be the union of Z' and Z''. Then it is easy to see that with this set Z, X' is an order-separable space with a natural topology.

Hence, we have proved the following theorem :

Theorem 4—Let $(X, \precsim)$ be a completely preordered second countable topological space with a natural topology. Then $(X, \precsim)$ can be embedded in an order-separable space with a natural topology.

Theorem 4 and Peleg's theorem imply Debreu's Theorem 2.

Conclusion

It has been shown above that both the classic Debreu theorems in utility theory are consequences of Peleg's theorem. Since Debreu first proved these theorems in 1954, much work has been done on simplifying and refining Debreu's methods. Many alternative proofs of Debreu's theorems have been subsequently discovered. For a discussion of these developments (see Mehta 1981). We may conclude by observing that Peleg's theorem offers us yet another method of proving the classic Debreu theorems.

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References

Debreu, G. (1954). Representation of a preference ordering by a numerical function in 'Decision Processes', (eds. R. Thrall, C. Coombs, and R. Davis), New York, Wiley, 159–66.

Lee, L. (1972). The theorems of Debreu and Peleg for ordered topological spaces. *Econometrica*, 1151–53.

Mehta, G. (1977). Topological ordered spaces and utility functions. *International Economic Review*, 779–82.

———(1983). Recent developments in utility theory. *Indian Economic J.*, (to appear).

Peleg, B. (1970). Utility functions for partially ordered topological spaces. *Econometrica*, 93–96.