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STRESS DISTRIBUTION IN AN ANISOTROPIC ELLIPTIC RING COMPRESSED ALONG THE MAJOR AXIS

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The stress distribution in an anisotropic elliptic ring under constant pressure along the major axis is discussed in this paper.

1. INTRODUCTION

The problem of an infinite anisotropic plate with an elliptic hole under prescribed pressure on the boundary was solved by Green and Zerna (1954). Ōkubo (1952) considered the case of a circular disc of anisotropic materials. The present paper deals with an elliptic ring of anisotropic material compressed along the major axis by a pair of equal and opposite forces acting on its outer boundary. This problem is solved using complex variable technique and the stress distributions are presented as curves for a particular anisotropic material. The problem discussed here actually represents the stress distribution in long elliptic hollow cylinder of an anisotropic material, when stressed by constant outside pressure on two generators symmetrically situated. It has direct application to the stress-concentration analysis around elliptic hole in an anisotropic plate and also to the reinforcement problem by anisotropic elliptic ring. It is believed that this problem has not been discussed by any author.

2. FORMULATION OF THE PROBLEM

The problem discussed here is that of an elliptic annular anisotropic ring under compression on the outer boundary by two equal and opposite forces F, F acting along the major axis; the inner boundary being kept stress-free. The material for consideration is assumed to be the region bounded by the confocal ellipses

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \ldots(1)$$

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1 \qquad \ldots(2)$$

which have two orthogonal axes of symmetry and where $|a| > |a'|$, $|b| > |b'|$. Considering σ_{xx}, σ_{yy}, τ_{xy} to represent the stresses in the plane of the annulus, we have (Milne-Thomson 1960)

$$\left.\begin{aligned}\sigma_{xx} &= -\operatorname{Re}\sum_{j=1,2} k^2{}_j\Phi_j(z_j)\\ \sigma_{yy} &= \operatorname{Re}\sum_{j=1,2}\Phi_j(z_j)\\ \tau_{xy} &= \operatorname{Im}\sum_{j=1,2} k_j\Phi_j(z_j)\end{aligned}\right\} \quad \ldots(3)$$

where Re and Im represent the real and imaginary parts of a complex function, and $z_j = x + ik_jy$; k_1^2 and k_2^2 ($k_1 > 0$, $k_2 > 0$, $k_1 \geqslant k_2$) being the roots of the characteristic equation

$$l_{11}k^4 - 2l_{61}k^3 + (2l_{12} + l_{66})k^2 - 2l_{26}k + l_{22} = 0. \quad \ldots(4)$$

For later conveniences, we consider the transformation

$$z = \cosh(\xi + i\eta)$$

from the orthogonal system to the curvilinear one, thereby causing transformations in the stress components. If $\sigma_{\xi\xi}$, $\sigma_{\eta\eta}$, $\tau_{\xi\eta}$ are the stress components in the curvilinear co-ordinate system, from the relations between the fundamental stress combinations, we obtain

$$\left.\begin{aligned}\sigma_{\xi\xi} - \sigma_{\eta\eta} - 2i\tau_{\xi\eta} &= -e^{-2i\gamma}\left[\operatorname{Re}\sum_{j=1,2}(1+k_j^2)\Phi_j(z_j) - 2i\operatorname{Im}\sum_{j=1,2} k_j\Phi_j(z_j)\right]\\ \sigma_{\xi\xi} + \sigma_{\eta\eta} &= \operatorname{Re}\sum_{j=1,2}(1-k_j^2)\,\Phi_j(z_j)\end{aligned}\right\} \quad \ldots(5)$$

where γ is the angle made by the normal to the curve at the point (ξ, η) with the x-axis, and is given by

$$\exp(2i\gamma) = \sinh(\xi + i\eta)/\sinh(\xi - i\eta) = P + iQ \quad \ldots(6)$$

where

$$\left.\begin{aligned}P &= (\cosh 2\xi \cos 2\eta - 1)/(\cosh 2\xi - \cos 2\eta)\\ Q &= \sinh 2\xi \sin 2\eta/(\cosh 2\xi - \cos 2\eta).\end{aligned}\right\} \quad \ldots(7)$$

For the outer ellipse, we reserve the transformations

$$z = c\cosh(\alpha + i\beta),\ z_j = x + ik_jy = c_j\cosh(\alpha_j + i\beta_j),\ j = 1,2 \quad \ldots(8)$$

and observe that, if $\alpha = s$ (a constant), $\alpha_j = s_j$ (a constant), $\beta = \beta_1 = \beta_2$ represent this boundary, and that

$$a = c\cosh s = c_j\cosh s_j,\ b = c\sinh s = (c_j/k_j)\sinh s_j. \quad \ldots(9)$$

Similar transformations

$$z = c\cosh(\alpha' + i\beta'),\ z_j = x + ik_jy = c_j'\cosh(\alpha_j' + i\beta_j'),\ j = 1,2 \quad \ldots(10)$$

on the inner ellipse show that, if $\alpha' = s'$ (a constant), $\alpha_j' = s_j'$ (a constant), $\beta' = \beta_1'$ $= \beta_2'$ represent this boundary, and that

$$a' = c \cosh s' = c_j' \cosh s_j', \quad b' = c \sinh s' = (c_j'/k_j) \sinh s_j'. \qquad \ldots(11)$$

3. Solution of the Problem

The conditions on the boundary are

$$\left.\begin{aligned} &\sigma_{\xi\xi} = \tau_{\xi\eta} = 0 \qquad \text{on } \xi = s',\ \beta' = \beta_1' = \beta_2' \\ &\sigma_{\xi\xi} = -(F/\pi)[1 + 2\sum_{n=1}^{\infty} \cos 2n\beta] \quad \text{on } \xi = s,\ \beta = \beta_1 = \beta_2 \\ &\tau_{\xi\eta} = 0. \end{aligned}\right\} \qquad \ldots(12)$$

For evaluation of the state of stress under these boundary conditions, let us assume that the potential function is represented as a power series,

$$\Phi_j(z_j) = \sum_{n=0}^{\infty} A_n^j \cosh 2n(\xi_j + i\eta_j) + \sum_{n=1}^{\infty} B_n^j \sinh 2n(\xi_j + i\eta_j) \qquad \ldots(13)$$

with the help of which (5) yield

$$\begin{aligned} 2\sigma_{\xi\xi} = \sum_{j=1,2} \Big[& -P(1+k_j^2) \sum_{n=0}^{\infty} A_n^j \cosh 2n\xi_j \cos 2n\eta_j \\ & -P(1+k_j^2) \sum_{n=1}^{\infty} B_n^j \sinh 2n\xi_j \cos 2n\eta_j \\ & + 2Qk_j \sum_{n=1}^{\infty} A_n^j \sinh 2n\xi_j \sin 2n\eta_j + 2Qk_j \sum_{n=1}^{\infty} B_n^j \cosh 2n\xi_j \sin 2n\eta_j \\ & + (1-k_j^2) \sum_{n=0}^{\infty} A_n^j \cosh 2n\xi_j \cos 2n\eta_j \\ & + (1-k_j^2) \sum_{n=1}^{\infty} B_n^j \sinh 2n\xi_j \cos 2n\eta_j \Big] \end{aligned} \qquad \ldots(14)$$

$$2\sigma_{\eta\eta} = \sum_{j=1,2}\Big[P(1+k_j^2)\sum_{n=0}^{\infty}A_n^j \cosh 2n\xi_j \cos 2n\eta_j$$

$$+P(1+k_j^2)\sum_{n=1}^{\infty}B_n^j \sinh 2n\xi_j \cos 2n\eta_j$$

$$-2Qk_j\sum_{n=1}^{\infty}A_n^j \sinh 2n\xi_j \sin 2n\eta_j$$

$$-2Qk_j\sum_{n=1}^{\infty}B_n^j \cosh 2n\xi_j \sin 2n\eta_j$$

$$+(1-k_j^2)\sum_{n=0}^{\infty}A_n^j \cosh 2n\xi_j \cos 2n\eta_j$$

$$+(1-k_j^2)\sum_{n=1}^{\infty}B_n^j \sinh 2n\xi_j \cos 2n\eta_j\Big] \qquad ...(15)$$

$$2\tau_{\xi\eta} = \sum_{j=1,2}\Big[2Pk_j\sum_{n=1}^{\infty}A_n^j \sinh 2n\xi_j \sin 2n\eta_j$$

$$+2Pk_j\sum_{n=1}^{\infty}B_n^j \cosh 2n\xi_j \sin 2n\eta_j$$

$$+Q(1+k_j^2)\sum_{n=0}^{\infty}A_n^j \cosh 2n\xi_j \cos 2n\eta_j$$

$$+Q(1+k_j^2)\sum_{n=1}^{\infty}B_n^j \sinh 2n\xi_j \cos 2n\eta_j\Big]. \qquad ...(16)$$

Using these expressions in (12) for the inner and the outer boundaries of the annulus, we obtain after some manipulations

$$\sum_{j=1,2}\Big[(c_j^2/c^2)\Big\{C_n^j \cosh 2ns_j' + D_n^j \sinh 2ns_j'\Big\}\Big] = 0,\ n \geqslant 1 \qquad ...(17)$$

$$\sum_{j=1,2}\left[(c_j^2/c^2)\left\{C_n^j \cosh 2(n+1)s_j + D_n^j \sinh 2(n+1)s_j'\right\}\right] = 0, \quad n \geqslant 1 \quad ...(18)$$

$$\sum_{j=1,2}\left[(c_j^2/c^2)\left\{C_n^j \cosh 2ns_j + D_n^j \sinh 2ns_j\right\}\right] = -2(F/\pi)(\cosh 2s-1), \quad n \geqslant 1 \quad ...(19)$$

$$\sum_{j=1,2}\left[(c_j^2/c^2)\left\{C_n^j \cosh 2(n+1)s_j + D_n^j \sinh 2(n+1)s_j\right\}\right] = 2(F/\pi) \times (\cosh 2s-1), \quad n \geqslant 1 \quad ...(20)$$

where

$$\left.\begin{aligned} &C_{n-1}^j = A_{n-1}^j - A_n^j, \quad D_{n-1}^j = B_{n-1}^j - B_n^j, \quad n > 1 \\ &C_0^j = 2A_0^j - A_1^j \end{aligned}\right\} \quad ...(21)$$

Solving for C_n^j, D_n^j, $j = 1,2$ from eqns. (17), (18), (19) and (20), we get

$$\left.\begin{aligned} C_n^1 &= \{T_1L_2F/(\pi\Delta)\}(L_1D_3\chi_2' - L_1D_4\chi_4' - L_2D_1 \sinh 2s_2')(\cosh 2s-1) \\ D_n^1 &= \{T_1L_2F/(\pi\Delta)\}(L_1D_4\chi_3 - L_1D_3\chi_1' + L_2D_2 \sinh 2s_2')(\cosh 2s-1) \\ C_n^2 &= \{T_2L_1F/(\pi\Delta)\}(L_2D_1\chi_1' - L_2D_2\chi_2' - L_1D_4 \sinh 2s_1')(\cosh 2s-1) \\ D_n^2 &= \{T_2L_1F/(\pi\Delta)\}(L_2D_2\chi_4' - L_2D_1\chi_3' + L_1D_3 \sinh 2s_1')(\cosh 2s-1) \end{aligned}\right\} \quad ...(22)$$

where

$$\left.\begin{aligned} T_1 &= 2c^2/c_1^2, \qquad T_2 = 2c^2/c_2^2 \\ L_1 &= c_1'^2/c_1^2, \qquad L_2 = c_2'^2/c_2^2 \end{aligned}\right\} \quad ...(23)$$

$$\left.\begin{aligned} \chi_1 &= \sinh 2(n+1)s_2 \cosh 2ns_1 - \sinh 2ns_2 \cosh 2(n+1)s_1 \\ \chi_2 &= \sinh 2(n+1)s_2 \sinh 2ns_1 - \sinh 2ns_2 \sinh 2(n+1)s_1 \\ \chi_3 &= \cosh 2(n+1)s_2 \cosh 2ns_1 - \cosh 2ns_2 \cosh 2(n+1)s_1 \\ \chi_4 &= \cosh 2(n+1)s_2 \sinh 2ns_1 - \cosh 2ns_2 \sinh 2(n+1)s_1 \end{aligned}\right\} \quad ...(24)$$

$$\left.\begin{aligned} \chi_1' &= \sinh (2n+1)s_2' \cosh 2ns_1' - \sinh 2ns_2' \cosh 2(n+1)s_1' \\ \chi_2' &= \sinh (2n+1)s_2' \sinh 2ns_1' - \sinh 2ns_2' \sinh 2(n+1)s_1' \\ \chi_3' &= \cosh 2(n+1)s_2' \cosh 2ns_1' - \cosh 2ns_2' \cosh 2(n+1)s_1' \\ \chi_4' &= \cosh (2n+1)s_2' \sinh 2ns_1' - \cosh 2ns_2' \sinh 2(n+1)s_1' \end{aligned}\right\} \quad ...(25)$$

$$\left.\begin{aligned} D_1 &= \sinh 2(n+1)s_1 + \sinh 2ns_1,\ D_2 = \cosh 2(n+1)s_1 + \cosh 2ns_1 \\ D_3 &= \cosh 2(n+1)s_2 + \cosh 2ns_2,\ D_4 = \sinh 2(n+1)s_2 + \sinh 2ns_2 \end{aligned}\right\} \quad \ldots(26)$$

$$\Delta = L_1^2 \sinh 2s_1' \sinh 2s_2 + L_1L_2(\chi_1\chi_1' - \chi_2\chi_3' - \chi_3\chi_2' + \chi_4\chi_1') + L_2^2 \sinh 2s_1 \sinh 2s_2'. \quad \ldots(27)$$

Once the constants C_n^j, D_n^j are calculated, A_n^j and B_n^j follow from the relations (21) and the stresses are obtained.

4. Numerical Example

We consider a ring whose boundaries are a set of two confocal ellipses such that

$$a = 1.1a',\ b = 3.8b',\ a' = 8.0b'$$

where a, b are the lengths of the semi-axes of the outer ellipse and a', b' are those of the inner one.

Keeping the inner boundary stress-free, the outer boundary is compressed diametrically along the major axis by a pair of equal and opposite forces F, F. It is supposed that the ring is made out of wooden materials and major axis of the set of confocal ellipses coincide with the direction of the fibres of the material. We have

$$k_1 = 2.307,\ k_2 = 1.053 \quad \ldots(28)$$

$$\left.\begin{aligned} &c = 7.9372539\,b',\ s = 0.4621290,\ s' = 0.1256570, \\ &c_1 = 0.7659793\,b',\ c_1' = 7.6601404\,b',\ s_1 = 3.1326 \\ &s_1' = 0.296793,\ c_2 = 7.8376525\,b',\ c_2' = 7.9303966\,b', \\ &s_2 = 0.490615,\ s_2' = 0.132393,\ L_1 = 100.00907, \\ &L_2 = 1.0238063,\ T_1 = 214.75163,\ T_2 = 2.0511552. \end{aligned}\right\} \quad \ldots(29)$$

The stress distribution at points $(8(1 + 0.01r)b', 0)$ and $(0, (1+ 0.28r)b')$, $r=1,2,\ldots,9$ of the ring are calculated.

Let C be the centre of the system of the given confocal ellipses, representing the boundaries which intercept the major and minor axes at points A, A' and B, B' respectively. In order to calculate the stresses at above noted points, we bring into picture the nine numbers of ellipses, which pass through these respective points and which belong to the same confocal ellipse system of which the boundaries of the ring are members.

The transformations corresponding to these ellipses be

$$z = c\cosh(\alpha^{(r)} + i\beta^{(r)}),\ z_j = c_j^{(r)}\cosh(\alpha_j^{(r)} + i\beta_j^{(r)}),\ r = 1,2,\ldots,9 \quad (30)$$

and the respective ellipses are represented by

$$\left.\begin{aligned}&\alpha^{(r)} = s^{(r)}(r \text{ numbers of constants}),\ \alpha_j^{(r)} = s_j^{(r)}\ (r \text{ numbers of constants})\\ &\beta^{(r)} = \beta_1^{(r)} = \beta_2^{(r)}\end{aligned}\right\} \quad \ldots(31)$$

and correspondingly,

$$\left.\begin{aligned}P^{(r)} &= (\cosh 2\alpha^{(r)}\cos 2\beta^{(r)} - 1)/(\cosh 2\alpha^{(r)} - \cos 2\beta^{(r)})\\ Q^{(r)} &= (\sinh 2\alpha^{(r)}\sin 2\beta^{(r)}/(\cosh 2\alpha^{(r)} - \cos 2\beta^{(r)}).\end{aligned}\right\} \quad \ldots(32)$$

At points of AA' of the ring, we have for $r = 1,2,\ldots,9$

$$\left.\begin{aligned}&P^{(r)} = 1,\ Q^{(r)} = 0,\quad \beta^{(r)} = \beta_1^{(r)} = \beta_2^{(r)} = 0\\ &s_j^{(r)} = \cosh^{-1}\{8(1+0.01r)/[64(1+0.01r)^2 - (1+0.28r)^2k_j^2]\},\ j=1,2\end{aligned}\right\} \quad \ldots(33)$$

and at points of BB' of the ring we have for $r = 1,2,\ldots,9$

$$\left.\begin{aligned}&P^{(r)} = -1,\ Q^{(r)} = 0,\ \beta^{(r)} = \beta_1^{(r)} = \beta_2^{(r)} = \pi/2\\ &s_j^{(r)} = \sinh^{-1}\{(1+0.28r)k_j/[64(1+0.01r)^2 - (1+0.28r)^2k_j^2]\},\ j = 1,2\end{aligned}\right\} \quad \ldots(34)$$

Using high speed computer the stresses with the help of the eqns. (32), (33) and (34) at $(8(1 + 0.01r)b', 0)$ of the major axis and at $(0, (1 + 0.28r)b')$ of the minor axis are calculated.

In Figs. 1 and 2 the stresses $(\sigma_{\xi\xi}/F) \times 10^5$ (parallel to y-axis) and $(\sigma_{\eta\eta}/F) \times 10^5$ (parallel to x-axis) at $(8(1 + 0.01r)b', 0)$ of the major axis and in Figs. 3 and 4 the

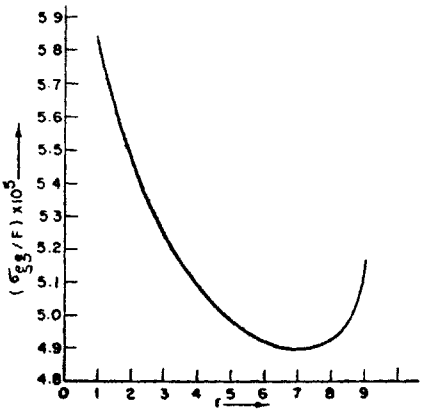

FIG. 1. $(\sigma_{\xi\xi}/F) \times 10^5$ (parallel to y-axis) versus points on major axis.

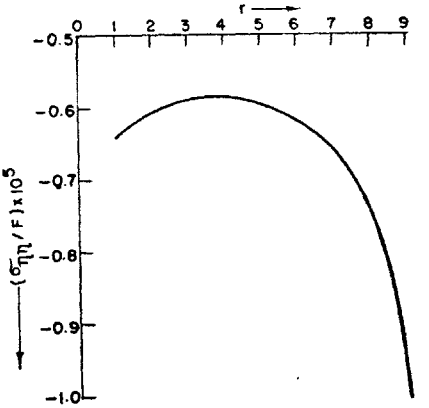

FIG. 2. $(\sigma_{\eta\eta}/F) \times 10^5$ (parallel to x-axis) versus points on major axis.

stresses $(\sigma_{\xi\xi}/F) \times 10^5$ (parallel to y-axis) and $(\sigma_{\eta\eta}/F) \times 10^5$ (parallel to x-axis) at $(0, (1 + 0.28r)b')$ of the minor axis are respectively plotted versus the values of γ.

And it is easily seen that at all these points $\tau_{\xi\eta} = 0$.

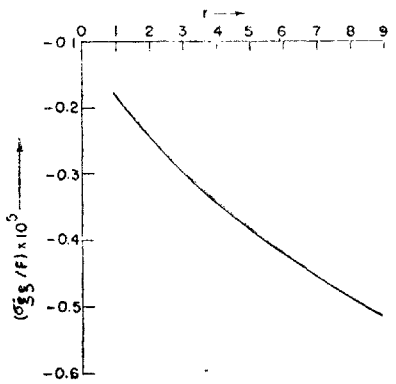

FIG. 3. $(\sigma_{\xi\xi}/F) \times 10^5$ (parallel to y-axis) versus points on minor axis.

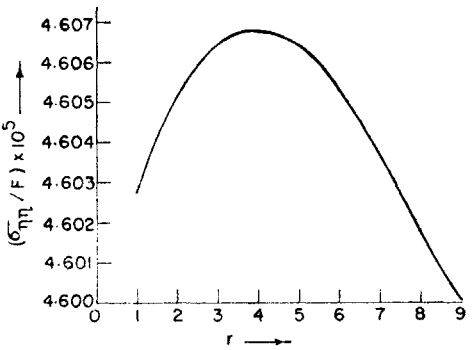

FIG. 4. $(\sigma_{\eta\eta}/F) \times 10^5$ (parallel to x-axis) versus points on minor axis.

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