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ON A SEQUENCE OF FOURIER COEFFICIENTS

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In the present paper a result regarding the $(N, p_n)\, C_1$ summability of a sequence $\{n\, B_n(x)\}$ has been obtained which generalizes the previous results of Sharma (1970) and Prasad (1981, 1983).

§ 1. Let Σa_n be a given infinite series with the sequence of partial sums $\{S_n\}$. Let $\{p_n\}$ be a sequence of constants, real or complex, such the $P_n = \sum_{k=0}^{n} p_k$, $p_{-1} = P_{-1} = 0$.

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k}\, S_k \qquad \ldots (1.1)$$

defines the sequence $\{t_n\}$ of the Nörlund means of the sequence $\{S_n\}$ generated by the sequence of constants $\{p_n\}$. The series Σa_n or the sequence $\{S_n\}$ is said to be summable by Nörlund means (Nörlund 1919, Woronoi 1932), or summable (N, p_n) to the limit S, if $\lim_{n \to \infty} t_n = S$.

The conditions of regularity of the method (N, p_n) defined by (1.1) are

$$\lim_{n \to \infty} \frac{p_n}{P_n} = 0 \qquad \ldots (1.2)$$

and

$$\sum_{k=0}^{n} |\, p_k \,| = O\,(|P_n|), \text{ as } n \to \infty. \qquad \ldots (1.3)$$

If $\{p_n\}$ is real and non-negative, (1.3) is automatically satisfied and then (1.2) is the necessary and sufficient condition for the regularity of the method (N, p_n) (Hardy 1967). In the special case in which

$$p_n = \frac{1}{n+1}$$

and therefore

$$P_n \sim \log n, \text{ as } n \to \infty$$

t_n reduces to the familiar harmonic mean of $\{S_n\}$ and if it is denoted by t_n', then the series Σa_n, or the sequence $\{S_n\}$, is said to be summable by harmonic means to the sum S, if $\lim_{n\to\infty} t_n' = S$ (Riesz 1924).

If the method of summability (N, p_n) is superimposed on Cesàro means of order one, another method of summability (N, p_n) C_1 is obtained. For $p_n = 1/(n+1)$, this method reduces to the summability $(N, 1/(n+1))$ C_1.

§ 2. Let f be a periodic function with period 2π and integrable in the sense of Lebesgue over an interval $(-\pi, \pi)$. Let the Fourier Series of f be

$$\tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x). \qquad \ldots(2.1)$$

The conjugate series of (2.1) is

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) = \sum_{n=1}^{\infty} B_n(x) \qquad \ldots(2.2)$$

We write

$$\psi(t) = f(x+t) + f(x-t) - l, \text{ where } l \text{ is a constant.}$$

$$\tau = \left[\frac{1}{t}\right], \text{ the integral part of } \frac{1}{t}.$$

$$\Psi(t) = \int_0^t |\psi(u)|\, du.$$

§ 3. The classical results of Mohanty and Nanda (1954) and Varshney (1959) were improved by many research workers like Sharma (1969, 1970), Lal (1971), Dwivedi (1971), Prasad (1981, 1982) and others in various directions. The object of the present paper is to study the (N, p_n) C_1 summability of a sequence of Fourier coefficients under less stringent conditions on the sequence $\{p_n\}$ and under more general conditions on the function. Namely, we prove the following :

Theorem—Let $\{p_n\}$ be a sequence such that $p_n > 0$, p_n ; and $p_n \to \infty$ and let $\lambda(t)$ be a positive integrable monotonic non-decreasing function such that

$$\int_1^n \frac{\lambda(u)}{u}\, du = O(P_n), \text{ as } n \to \infty. \qquad \ldots(3.1)$$

If

$$\Psi(t) = o\left(\frac{t\,\lambda\left(\frac{1}{t}\right)}{P_\tau}\right), \text{ as } t \to +0 \qquad \ldots(3.2)$$

then the sequence $\{n\,B_n(x)\}$ is summable (N, p_n) C_1 to the value l/π, where $P_n = p_0 + p_1 + p_2 + \ldots + p_n$ and τ is the integral part of $1/t$.

§ 4. We require the following lemmas to prove our theorem :

Lemma 1 (McFadden 1942)—If $\{p_n\}$ is non-negative and non-increasing, then for $0 \leqslant a < b \leqslant \infty$, $0 < t \leqslant \pi$ and any n

$$\left| \sum_{k=a}^{b} p_k\, e^{i(n-k)t} \right| \leqslant A\, P_\tau$$

where A is an absolute constant.

Lemma 2 (Prasad 1981)—If $0 \leqslant t \leqslant 1/n$, then

$$|\,Q_n(t)\,| = \frac{1}{\pi P_n} \sum_{k=1}^{n} p_{n-k} \left(\frac{\sin kt}{kt^2} - \frac{\cos kt}{t} \right) = O(n).$$

Lemma 3 (Dwivedi 1971)—For $0 < t \leqslant \pi$

$$|\,Q_n(t)\,| = O\left(\frac{P_\tau}{t\,P_n}\right).$$

§ 5. *Proof of the theorem*—If we denote the $(C, 1)$ transform of the sequence $\{n\,B_n(x)\}$ by t_n, we have, after Mohanty and Nanda (1954), that

$$t_n - \frac{l}{\pi} = \frac{1}{\pi} \sum_{r=1}^{n} r\,B_r(x) - \frac{l}{\pi}$$

$$= \frac{1}{\pi} \int_0^\pi \psi(t) \left\{ \frac{1}{4n} \frac{\sin nt}{\sin^2 t/2} - \frac{1}{2} \frac{\cos nt}{\tan \frac{1}{2}t} \right\} dt$$

$$= \frac{1}{2\pi} \int_0^\pi \psi(t) \sin nt\, dt + o(1)$$

(*equation continued on p.* 1248)

$$= \frac{1}{\pi} \int_0^{\pi} \psi(t) \left[\frac{\sin nt}{nt^2} - \frac{\cos nt}{t} \right] dt + o(1)$$

as the second integral becomes $o(1)$, by Riemann-Lebesgue theorem.

Since the method of summability under consideration is regular, we have to show that under the conditions of the theorem

$$I \equiv \int_0^{\pi} \frac{\psi(t)}{\pi P_n} \sum_{k=1}^{n} p_{n-k} \left[\frac{\sin kt}{kt^2} - \frac{\cos kt}{t} \right] dt$$

$$= o(1), \text{ as } n \to \infty.$$

Let us write

$$Q_n(t) \equiv \frac{1}{\pi P_n} \sum_{k=1}^{n} p_{n-k} \left[\frac{\sin kt}{kt^2} - \frac{\cos kt}{t} \right].$$

Therefore,

$$I = \int_0^{\pi} \psi(t)\, Q_n(t)\, dt$$

$$= \left(\int_0^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{\pi} \right) \psi(t)\, Q_n(t)\, dt$$

$$= I_1 + I_2 + I_3 \text{ say, where } 0 < \delta < \pi.$$

Now

$$I_1 = \int_0^{1/n} \psi(t)\, Q_n(t)\, dt$$

$$= O\left[n \int_0^{1/n} |\psi(t)|\, dt \right], \text{ by Lemma 2.}$$

$$= O\left[n.\, o\left\{ \frac{\lambda(n)}{n P_n} \right\} \right]$$

$$= o(1), \text{ as } n \to \infty.$$

$$I_2 = \int_{1/n}^{\delta} \psi(t)\, Q_n(t)\, dt$$

$$= O\left(\frac{1}{P_n} \right) \int_{1/n}^{\delta} |\psi(t)| \frac{P_r}{t}\, dt$$

(equation continued on p. 1249)

$$= O\left(\frac{1}{P_n}\right).\, o\left\{\Psi(t).\,\frac{P_\nu}{t}\right\}_{1/n}^{\delta} + O\left(\frac{1}{P_n}\right)\int_{1/n}^{\delta}\Psi(t).\,\frac{P_\nu}{t^2}dt$$

$$+ O\left(\frac{1}{P_n}\right)\int_{1/n}^{\delta}\Psi(t).\,\frac{1}{t}\,d\,P_\nu.$$

$$= I_{2\cdot1} + I_{2\cdot2} + I_{2\cdot3}, \text{ say.}$$

$$I_{2\cdot1} = O\left(\frac{1}{P_n}\right) o\left\{\lambda\left(\frac{1}{t}\right)\right\}_{1/n}^{\delta} = o\left(\frac{1}{P_n}\right) O(\lambda(n)) = o(1), \text{ as } n \to \infty.$$

$$I_{2\cdot2} = O\left(\frac{1}{P_n}\right)\int_{1/n}^{\delta}\Psi(t).\,\frac{P_\nu}{t^2}\,dt$$

$$= o(1) + O\left(\frac{1}{P_n}\right)\left[\sum_{k=1}^{n-1}\int_{k}^{k+1}\Psi\left(\frac{1}{u}\right)P_{[u]}\,du\right].$$

But

$$\int_{k}^{k+1}\Psi\left(\frac{1}{u}\right)P_{[u]}\,du \leqslant \Psi\left(\frac{1}{k}\right)P_k = o\left(\frac{\lambda(k)}{k}\right).$$

So,

$$I_{2\cdot2} = o(1) + O\left(\frac{1}{P_n}\right)\sum_{k=1}^{n-1} o\left(\frac{\lambda(k)}{k}\right)$$

$$= o(1) + O\left(\frac{1}{P_n}\right) o\left(\int_{1}^{n}\frac{\lambda(k)}{k}\,du\right)$$

$$= o(1) + o(1) = o(1), \text{ as } n \to \infty.$$

$$I_{2\cdot3} = O\left[\frac{1}{P_n}\int_{1/n}^{\delta}\Psi(t)\,\frac{d\,P_\nu}{t}\right]$$

$$= O\left[\frac{1}{P_n}\int_{1/\delta}^{n}\Psi\left(\frac{1}{u}\right)u\,d\,P_{[u]}\right]$$

(*equation continued on p.* 1250)

$$= O(1) + O\left[\frac{1}{P_n}\sum_{k=1}^{n} kp_k.\ \Psi\left(\frac{1}{k}\right)\right]$$

$$= o(1) + O\left[\frac{1}{P_n}\sum_{k=1}^{n-1} P_k.\,\Psi\left(\frac{1}{k}\right)\right]$$

$$= o(1) + O\left[\frac{1}{P_n}\sum_{k=1}^{n-1} o\left(\frac{\lambda(k)}{k}\right)\right]$$

$$= o(1) + O\left[\frac{1}{P_n}.\ o\left(\int_1^n \frac{\lambda(u)}{u}\,du\right)\right]$$

$$= o(1) + o(1), \text{ as } n \to \infty$$

$$= o(1).$$

Since the method of summation is regular, we have by Riemann-Lebesgue theorem

$$I_3 = o(1), \text{ as } n \to \infty.$$

Thus the proof of the theorem is complete.

Remarks: (1) If we take $\lambda(t)$ as a unity function, our theorem reduces to the following result of Sharma (1971).

Theorem—Let $\{p_n\}$ be a monotonic non-increasing sequence of real constants such that

(i) $P_n = \sum_{k=0}^{n} p_k$, $P_n \to \infty$, as $n \to \infty$,

(ii) $\log n = O(P_n)$, as $n \to \infty$,

(iii) $\int_0^t |\psi(u)|\,du = o\left(\frac{t}{P_\tau}\right)$, as $t \to 0$, where $\tau = \left[\frac{1}{t}\right]$.

Then the sequence $\{n B_n(x)\}$ is summable (N, p_n) C_1 to the value l/π.

(2) If we take $\lambda(t)$ as a positive monotonic non-decreasing function, then our theorem reduce to the following theorem of Prasad (1983).

Theorem—Let (N, p_n) be a regular Nörlund method defined by a sequence $\{p_n\}$ of real, positive, monotonic non-increasing coefficients such that

(i) $P_n = \sum_{k=0}^{n} p_k$, $P_n \to \infty$, as $n \to \infty$,

(ii) $\lambda(n) \log n = O(P_n)$, as $n \to \infty$

(iii) $\Psi(t) = o\left(\frac{\lambda\left(\frac{1}{t}\right) t}{P_{\frac{1}{t}}}\right)$, as $t \to +0$

then the sequence $\{n B_n(x)\}$ is summable (N, p_n) C_1 to the value $1/\pi$, where $\lambda(t)$ is a positive monotonic non-decreasing function.

(3) If we take $\lambda(t) = P(t)/\alpha(t)$, then our present theorem reduces to the following theorem of Prasad (1981).

Theorem—Let $p(u)$ be monotonic decreasing and strictly positive for $u \geq 0$.

Let $p_n = p(n)$ and $P(u) = \int_0^u p(x)\,dx$, such that

$$P(u) \to \infty, \text{ as } u \to \infty.$$

Let $\alpha(t)$ be a positive non-decreasing function of t. If

$$\Psi(t) \equiv \int_0^t |\psi(u)|\,du = o\left(\frac{t}{\alpha\left(\frac{1}{t}\right)}\right), \text{ as } t \to +0,$$

then a sufficient condition that the sequence $\{n B_n'(x)\}$ be summable (N, p_n) C_1 to the value $1/\pi$, is that

$$\int_1^u \frac{P(t)}{t\,\alpha(t)}\,dt = O(P(u)), \text{ as } u \to \infty.$$

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