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RELATIVE CONTROLLABILITY OF A CLASS OF NONLINEAR SYSTEMS WITH DELAY IN CONTROL

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Sufficient conditions for global relative controllability of general nonlinear time varying systems with time varying delay in control having implicit derivative are given. The results are obtained by using the measure of noncompactness of a set and the Darbo fixed point theorem.

1. INTRODUCTION

Controllability of nonlinear systems with implicit derivative has been studied by Dacka (1980). The method has been extended to nonlinear systems with time varying delay and implicit derivative by Dacka (1981, 1982). In this paper we shall further extend the technique to study the controllability of general nonlinear systems with implicit derivative and time varying delay. Using the measure of noncompactness of a set and Darbo's fixed point theorem, sufficient conditions for global relative controllability are derived. The results are a generalization of results given by Klamka (1975) and Dacka (1982).

The measure of noncompactness for metric spaces has been introduced by Kuratowski. We shall consider here only the case of Banach spaces. Let $(X, \|\cdot\|)$ be a Banach space and E a bounded subset of X. In this paper the following definition of the measure of noncompactness of a set E is used (Dacka 1980 and Sadovskii 1972).

$\mu(E) = \inf \{r > 0;$ E can be covered by a finite number of balls whose radii are smaller than $r\}$.

The following version of Darbo's fixed point theorem, being a generalisation of Schauder's fixed point theorem shows the usefulness of the measure of noncompactness (Sadovskii 1972).

"If S is a nonempty bounded closed convex subset of X and $P : S \to S$ is a continuous mapping such that for any $E \subset S$ we have

$$\mu(PE) \leqslant k\, \mu(E) \qquad \ldots(1)$$

where k is a constant, $0 \leqslant k < 1$, then P has a fixed point".

Let $C_n[t_0, t_1]$ be the space of continuous R^n valued functions on $[t_0, t_1]$ with the norm

$$\| x \| = \max \{ | x_i(t) | : i = 1, 2, \ldots, n, t \in [t_0, t_1] \}.$$

Denote $C_n^1[t_0, t_1]$ as the space of continuously differentiable R^n valued functions on $[t_0, t_1]$ with the norm

$$\| x \|_{C_n^1} = \| x \|_{C_n} + \| Dx \|_{C_n}$$

where $D = d/dt$ denotes the operation of differentiation.

For the space of continuous functions $C_n[t_0, t_1]$ the measure of noncompactness of a set E is given by

$$\mu(E) = \tfrac{1}{2} w_0(E) = \tfrac{1}{2} \lim_{h \to 0^+} w(E, h) \qquad \ldots(2)$$

where $w(E, h)$ is the common modulus of continuity of the functions which belong to the set E; that is

$$w(E, h) = \sup_{x \in E} [\sup | x(t) - x(s) | : | t - s | \leqslant h] \qquad \ldots(3)$$

where in the space of continuosuly differentiable functions $C_n^1[t_0, t_1]$ we have

$$\mu(E) = \tfrac{1}{2} w_0(DE) \qquad \ldots(4)$$

where

$$DE = \{\dot{x} : x \in E\}.$$

2. Statement of the Problem

Consider the following nonlinear time varying systems with time variable delay in control, represented by the equation

$$\dot{x} = A(t, x, \dot{x})x + B(t, x, \dot{x})\, u + C(t, x, \dot{x})\, u(h(t)) + f(t, x, \dot{x}) \qquad \ldots(5)$$

where x and f are n-vectors, u is an m-vector, $A(t, x, \dot{x})$ is an $n \times n$ matrix and $B(t, x, \dot{x})$, $C(t, x, \dot{x})$ are $n \times m$ matrices respectively. Assume that the elements $a_{ik}(t, x, \dot{x})$, $b_{ik}(t, x, \dot{x})$ $c_{ik}(t, x, \dot{x})$ of A, B, C respectively and $f(t, x, \dot{x})$ are continuous functions and fulfill the following conditions :

$$\left.\begin{aligned} & | a_{ik}(t, x, y) | \leqslant M \\ & | b_{ik}(t, x, y) | \leqslant N \\ & | c_{ik}(t, x, y) | \leqslant R \\ & | f(t, x, y) | \leqslant K \text{ for } t \in [t_0, t_1] \text{ and } x, y \in R^n \end{aligned}\right\} \qquad \ldots(6)$$

where M, N, R and K are some positive constants. Further, for every $y, \bar{y} \in R^n$ and $x \in R^n$, $t \in [t_0, t_1]$

$$\left.\begin{aligned} | a_{ik}(t, x, y) - a_{ik}(t, x, \bar{y}) | &\leqslant \frac{k_1}{n^2} | y - \bar{y} | \\ | b_{ik}(t, x, y) - b_{ik}(t, x, \bar{y}) | &\leqslant \frac{k_2}{nm} | y - \bar{y} | \\ | c_{ik}(t, x, y) - c_{ik}(t, x, \bar{y}) | &\leqslant \frac{k_3}{nm} | y - \bar{y} | \\ | f(t, x, y) - f(t, x, \bar{y}) | &\leqslant k_4 | y - \bar{y} | \end{aligned}\right\} \quad (7)$$

where k_1, k_2, k_3 and k_4 are positive constants such that $0 \leqslant k_i < 1/4$. Define the norm of a continuous $n \times m$ matrix valued function $W(t)$ by

$$\| W(t) \| = \max_i \sum_{j=1}^{m} \max_{t \in [t_0, t_1]} | w_{ij}(t) |$$

where w_{ij} are elements of W.

Let the function $h : [t_0, t_1] \to R$ be absolutely continuous, strictly increasing on $[t_0, t_1]$ and satisfying the following condition :

$$h(t) \leqslant t \quad \text{for} \quad t \in [t_0, t_1]. \qquad \ldots(8)$$

Let us introduce the time lead function $r(s)$ such that

$$r(h(t)) = t \quad \text{for} \quad t \in [t_0, t_1]. \qquad \ldots(9)$$

The following definitions related to the controllability of systems with delays are assumed (Klamka 1975).

Definition 1 — The set $y(t) = \{x(t), w(t, s)\}$ where $w(t, s) = u(s)$ for $s \in [h(t), t)$ is said to be the complete state of the system (5) at time t.

Definition 2—The system (1) is said to be globally relatively controllable on $[t_0, t_1]$ if for every complete state $y(t_0)$ and every vector $x_1 \in R^n$ there exists a control $u(t)$ defined on $[t_0, t_1]$ such that the corresponding trajectory of the system (5) satisfies $x(t_1) = x_1$.

For each $z \in C_n^1[t_0, t_1]$, consider the following linear system

$$\dot{x} = A(t, z, \dot{z})\, x + B(t, z, \dot{z})\, u + C(t, z, \dot{z})\, u(h(t)) + f(t, z, \dot{z}). \qquad \ldots(10)$$

The solution of the system (10) with the initial complete state $y(t_0)$ is of the form (Klamka 1975),

$$x(t) = \varphi(t, t_0; z, \dot{z}) \{x(t_0) + \int_{t_0}^{h(t)} [\varphi(t_0, s; z, \dot{z})\, B(s, z, \dot{z})$$

(*equation continued on p.* 1330)

$$+ \varphi(t_0, r(s); z, \dot{z})\, C(r(s), z, \dot{z})\, \dot{r}(s)]\, u(s)\, ds$$

$$+ \int_{h(t)}^{t} \varphi(t_0, s; z, \dot{z}) B(s, z, \dot{z})\, u(s)\, ds + \int_{t_0}^{t} \varphi(t_0, s; z, \dot{z}) f(s, z, \dot{z})\, ds$$

$$+ \int_{h(t_0)}^{t_0} \varphi(t_0, r(s); z, \dot{z})\, C(r(s), z, \dot{z})\, w(t_0, s)\, \dot{r}(s)\, ds\} \qquad \ldots(11)$$

where $\varphi(t, t_0; z, \dot{z})$ is the fundamental matrix of the system $\dot{x} = A(t, z, \dot{z})\, x$ with $\varphi(t_0, t_0; z, \dot{z}) = I$. Assume that $t_0 > r(t_0)$ and define

$$W(t_0, t; z, \dot{z}) = \int_{t_0}^{h(t)} [\varphi(t_0, s; z, \dot{z})\, B(s, z, \dot{z}) + \varphi(t_0, r(s); z, \dot{z})$$

$$\times C(r(s), z, \dot{z})\, \dot{r}(s)]\, [\varphi(t_0, s; z, \dot{z})\, B(s, z, \dot{z})$$

$$+ \varphi(t_0, r(s); z, \dot{z})\, C(r(s), z, \dot{z})\, \dot{r}(s)]'\, ds$$

$$+ \int_{h(t)}^{t} \varphi(t_0, s; z, \dot{z})\, B(s, z, \dot{z})\, B'(s, z, \dot{z})\, \varphi'(t_0, s; z, \dot{z}) ds$$

$$\ldots(12)$$

and

$$p(y(t_0), x(t); z, \dot{z}) = \varphi(t_0, t; z, \dot{z})\, x(t) - x(t_0)$$

$$- \int_{t_0}^{t} \varphi(t_0, t; z, \dot{z})\, f(s, z, \dot{z})\, ds - \int_{h(t_0)}^{t_0}$$

$$\times \varphi(t_0, r(s); z, \dot{z})\, C(r(s), z, \dot{z})\, w(t_0, s)\, \dot{r}(s)\, ds \quad \ldots(13)$$

here the prime denotes the matrix transpose.

Remark : Since very weak regularity conditions have been imposed on function f, we need not get a unique response for a given control u. It means that there exists a control u for which atleast one solution reaches the final state x_1. Suppose f satisfies the Lipschitz condition with respect to the vector x, the response is uniquely determined by any control.

3. Controllability Results

Theorem—Given the system (5) with conditions (6) to (9). Assume that $t_1 > r(t_0)$ and

$$\inf_{z \in C_n^1[t_0, t_1]} \det W(t_0, t_1; z, \dot{z}) > 0 \qquad \ldots(14)$$

where det W denotes the determinant of the matrix W. Then the system (5) is globally relatively controllable on $[t_0, t_1]$.

PROOF : Define the nonlinear operator T of the Banach space $C_n^1[t_0, t_1]$ by the formula

$$T(z)(t) = \varphi(t, t_0; z, \dot{z}) \{x(t_0) + \int_{t_0}^{h(t)} [\varphi(t_0, s; z, \dot{z}) B(s, z, \dot{z}) + \varphi(t_0, r(s); z, \dot{z}) C(r(s), z, \dot{z}) \dot{r}(s)] u(s)\, ds + \int_{h(t_0)}^{t} \varphi(t_0, s; z, \dot{z}) B(s, z, \dot{z}) u(s)\, ds + \int^{t} \varphi(t_0, s; z, \dot{z}) \times f(s, z, \dot{z})\, ds + \int_{h(t_0)}^{t_0} \varphi(t_0, r(s); z, \dot{z}) C(r(s), z, \dot{z}) \times w(t_0, s) \dot{r}(s)\, ds\} \quad \ldots(15)$$

where the control $u(t)$ for $t \in [t_0, t_1]$ is given by

$$u(t) = \begin{cases} [\varphi(t_0, t; z, \dot{z}) B(t, z, \dot{z}) + \varphi(t_0, r(t); z, \dot{z}) C(r(t), z, \dot{z}) \dot{r}(t)]' \\ W^{-1}(t_0, t; z, \dot{z}) p(y(t_0), x_1; z, \dot{z}) \quad \text{for} \quad t \in [t_0, h(t_1)] \\ B'(t, z, \dot{z}) \varphi'(t_0, t; z, \dot{z}) W^{-1}(t_0, t; z, \dot{z}) p(y(t_0), x_1; z, \dot{z}) \\ \qquad \text{for} \quad t \in [h(t_1), t_1) \end{cases} \quad \ldots(16)$$

where $y(t_0)$ and x_1 are chosen arbitrarily. Substituting (16) into (15), we get

$$T(z)(t) = \varphi(t, t_0; z, \dot{z}) \{x(t_0) + \int_{h(t_0)}^{t_0} \varphi(t_0, r(s); z, \dot{z}) C(r(s), z, \dot{z}) \times w(t_0, s) \dot{r}(s)\, ds + \int_{t_0}^{h(t)} [\varphi(t_0, s; z, \dot{z}) B(s, z, \dot{z}) + \varphi(t_0, r(s); z, \dot{z}) C(\dot{r}(s), z, \dot{z}) \dot{r}(s)] [\varphi(t_0, s; z, \dot{z}) \times B(s, z, \dot{z}) + \varphi(t_0, r(s); z, \dot{z}) C(r(s), z, \dot{z}) \dot{r}(s)]' W^{-1}(t_0, t_1; z, \dot{z}) p(y(t_0), x_1; z, \dot{z})\, ds + \int_{h(t)} \varphi(t_0, s; z, \dot{z}) B(s, z, \dot{z}) B'(s, z, \dot{z}) \varphi'(t_0, s; z, \dot{z}) W^{-1}(t_0, t_1; z, \dot{z}) p(y(t_0), x_1; z, \dot{z})\, ds + \int_{t_0} \varphi(t_0, s; z, \dot{z}) f(s, z, \dot{z}). \quad \ldots(17)$$

Since all the functions involved in the definition of the operator T are continuous it is easy to show that

$$\| T(z_1)(t) - T(z_2)(t) \|_{C_n^1} < \epsilon \text{ whenever } \| z_1 - z_2 \|_{C_n^1} < \delta$$

that is $\| z_1 - z_2 \|_{C_n} < \delta_1$ and $\| \dot{z}_1 - \dot{z}_2 \|_{C_n} < \delta_2$ which implies that T is continuous. Further T maps the Banach space $C_n^1[t_0, t_1]$ into itself.

Let us consider the closed and convex subset of $C_n^1[t_0, t_1]$ defined by

$$H = [z : z \in C_n^1[t_0, t_1], \| z \| \leqslant N_1, \| Dz \| \leqslant N_2]$$

where the positive real constants N_1 and N_2 are defined by

$$\begin{aligned} N_1 = \exp(nM(t_1 - t_0)) \{ & | x(t_0) | + K(t_1 - t_0) \exp(nM(t_1 - t_0)) \\ & + nmRLL_0 \exp(nM(t_1 - t_0))(t_0 - h(t_0)) \\ & + L_1 [(t_1 - t_0) nmN^2 \exp(2nM(t_1 - t_0)) + (h(t_1) - t_0) \\ & [2 nmNRL_0 \exp(nM(t_1 + h(t_1) - \gamma(t_0) - t_0)) \\ & + nmR^2 L_0^2 \exp(2nM(t_1 - r(t_0)))]]\} \end{aligned} \quad \ldots(18)$$

with

$$L_0 = \| \dot{r}(t) \| \quad \ldots(19)$$

$$L = \max_{r(t_0) \leqslant t \leqslant t_0} | w(t_0, t) | \quad \ldots(20)$$

$$\begin{aligned} L_1 = \sup_{z \in C_n^1[t_0, t_1]} \| W^{-1}(t_0, t_1, z, \dot{z}) \| [& | x_1 | \exp(nM(t_1 - t_0)) \\ & + | x(t_0) | + nmRLL_0 (t_0 - r(t_0)) \exp(nM(r(t_0)) - (t_0)) \\ & + K(t_1 - t_0) \exp(nM(t_1 - t_0))]. \end{aligned} \quad \ldots(21)$$

and

$$N_2 = n^2 MN_1 + nm(N + R) L_2 + K \quad \ldots(22)$$

with

$$L_2 = L_1 [nmN \exp(nM(t_1 - t_0)) + L_0 nmR \exp(nM(t_1 - r(t_0)))] \quad \ldots(23)$$

and

$$0 \leqslant k_1 N_1 < 1/4, \; 0 \leqslant k_2 L_2 < 1/4, \; 0 \leqslant k_3 L_2 < 1/4. \quad \ldots(24)$$

The operator T maps H onto itself. As easily seen, all the functions $T(z)(t)$ in $C_n^1[t_0, t_1]$ with $z \in H$ are equicontinuous, since they have uniformly bounded derivatives. Now we shall find an estimate of the modulus of continuity of the functions $DT(z)$ for $t, s \in [t_0, t_1]$

$$
\begin{aligned}
| DT(z)(t) - DT(z)(s) | \leqslant & | A(t, z(t), \dot{z}(t)) T(z)(t) \\
& - A(t, z(t), \dot{z}(s)) T(z)(t) | \\
& + | A(t, z(t), \dot{z}(s)) T(z)(t) - A(s, z(s), \dot{z}(s)) T(z)(s) | \\
& + | B(t, z(t), \dot{z}(t)) u(t) - B(t, z(t), \dot{z}(s)) u(t) | \\
& + | B(t, z(t), \dot{z}(s)) u(t) - B(s, z(s), \dot{z}(s)) u(s) | \\
& + | C(t, z(t), \dot{z}(t)) u(h(t)) - C(t, z(t), \dot{z}(s)) u(h(t)) | \\
& + | C(t, z(t), \dot{z}(s)) u(h(t)) - C(s, z(s), \dot{z}(s)) u(h(s)) | \\
& + | f(t, z(t), \dot{z}(t)) - f(t, z(t), \dot{z}(s)) | \\
& + | f(t, z(t), \dot{z}(s)) - f(s, z(s), \dot{z}(s)) | . \qquad \ldots(25)
\end{aligned}
$$

Following the technique of Dacka (1980), the first two terms of the right-hand side of inequality (25) can be estimated as $k_1 N_1 | \dot{z}(t) - \dot{z}(s) | + \beta_0 (| t - s |)$, where β_0 is a non negative function such that $\lim_{h \to 0^+} \beta_0(h) = 0$. In the same manner we find that the next two terms consecutively can be made as

$$k_2 L_2 | \dot{z}(t) - \dot{z}(s) | + \beta_1 (| t - s |), k_3 L_2 | \dot{z}(t) - \dot{z}(s) | + \beta_2 (| t - s |)$$

and $k_4 | \dot{z}(t) - \dot{z}(s) | + \beta_3 (| t - s |)$ respectively. Setting $k = k_1 N_1 + k_2 L_2 + k_3 L_2 + k_4$ and $\beta = \beta_0 + \beta_1 + \beta_2 + \beta_3$, we finally obtain

$$| DT(z)(t) - DT(z)(s) | \leqslant k | \dot{z}(t) - \dot{z}(s) | + \beta (| t - s |) \qquad \ldots(26)$$

hence we conclude that for any set $E \subset H$.

$$\mu(TE) \leqslant k \mu(E).$$

Consequently by the Darbo fixed point theorem the operator T has at least one fixed point therefore there exist a function $z^+ \in C_n^1 [t_0, t_1]$ such that

$$x^+(t) = z^+(t) = T(z^+(t)). \qquad \ldots(27)$$

Differentiating with respect to t, we easily verify that $x(t)$ given by (27) is a solution of the system (5) for the control $u(t)$ given by (16). The control steers the system (5) from the initial complete state $y(t_0)$ to $x_1 \in R^n$, on the interval $[t_0, t_1]$ and, since $y(t_0)$, and x_1 have been chosen arbitrarily then by Definition 2 the system (5) is globally relatively controllable on $[t_0, t_1]$.

In the case when the function $f(t, x, \dot{x}) = 0$ and $t_0 \leqslant t_1 \leqslant r(t_0)$, the global relative controllability of the system (5) does not depend on the delay and in addition A and B are independent of $\dot{x}$, the sufficient conditions for global relative controllability is of the form given by Davison and Kunze (1970).

Corollary — Let $h(t) = t - h$, $h > 0$. Then the system (5) is globally relatively on controllable on $[t_0, t_1]$ if

$$\inf_{z \in C_n^1 [t_0, t_1]} \det W_1 (t_0, t_1; z, \dot{z}) > 0 \qquad \ldots(28)$$

where

$$W_1 (t_0, t_1; z, \dot{z}) = \int_{t_0}^{t_1 - h} [\varphi (t_0, s; z, \dot{z}) B (s, z, \dot{z}) + \varphi (t_0, s+h; z, \dot{z}) \times C (s+h, z, \dot{z})] [\varphi (t_0, s; z, \dot{z}) B (s, z, \dot{z}) + \varphi (t_0, s+h; z, \dot{z}) C (s+h, z, \dot{z})]' ds + \int_{t_1 - h}^{t_1} \varphi (t_0, s; z, \dot{z}) B (s, z, \dot{z}) B' (s, z, \dot{z}) \varphi' (t_0, s; z, \dot{z}) ds. \qquad \ldots(29)$$

PROOF : If $h(t) = t - h$, then

$$r(t) = t + h, \; \dot{r}(t) = 1 \quad \text{for } t \in [t_0, t_1]. \qquad \ldots(30)$$

Substituting (30) into (12) and using Theorem 1 it is easily verified that the system (5) is globally relatively controllable on $[t_0, t_1]$ if the inequality (28) holds.

4. CONCLUSION

Using the measure of noncompactness of a set and the Darbo fixed point theorem sufficient conditions have been established for global relative controllability of non linear time varying systems with time variable delay in control and implicit derivative. The special case when the delay is constant has also been considered.

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