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ON A NEW SERIES TRANSFORM AND ITS CONVOLUTION

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Following the work of Gould (1964), we introduce here a new series transform and prove an interesting theorem on the convolution of this transform.

1. INTRODUCTION

During the course of our study of a general class of polynomials $\{f_n^c(x, y, r, m)\}$ which are defined [Singhal and Savita Kumari (1980)] by means of the generating relation

$$(1 + yt^m)^{-c} G\left[\frac{xt}{(1 + yt^m)^r}\right] = \sum_{n=0}^{\infty} f_n^c (x, y, r, m)\, t^n \qquad \ldots(1.1)$$

where

$$G[z] = \sum_{n=0}^{\infty} \gamma_n z^n \qquad \ldots(1.2)$$

m is a positive integer and other parameters are unrestricted in general, we obtained a new series transform (Singhal and Savita Kumari 1982),

$$F(n) = \sum_{k=0}^{[n/m]} y^k \binom{p-\lambda n + \lambda mk}{k} f(n-mk) \qquad \ldots(1.3)$$

where p and λ are arbitrary parameters.

For the sake of simplicity we shall denote the relation (1.3) symbolically as

$$F(n) = S_p [f(n)]. \qquad \ldots(1.4)$$

As was pointed out in Singhal and Savita Kumari (1982), (1.3) finds wide applications in the study of classical orthogonal polynomials and their various generalizations e.g. those given in Gould (1965), Jain (1967), Panda (1977).

The object of this paper is to show that the transform defined by (1.3) possesses an interesting property which we state in the following form :

Theorem—If

$$F(n) = S_p [f(n)] \qquad \ldots(1.5)$$

$$G(n) = S_q [g(n)] \qquad \ldots(1.6)$$

and if, $(F * G)(n)$ and $(f * g)(n)$ stand for the series convolutions defined by

$$(F * G)(n) = \sum_{j=0}^{n} F(j)\, G(n-j) \qquad \ldots(1.7)$$

$$(f * g)(n) = \sum_{j=0}^{n} f(j)\, g(n-j) \qquad \ldots(1.8)$$

then

$$(F * G)(n) = S_{p+q} [(f * g)(n)] \qquad \ldots(1.9)$$

that is, the convolution of the series transforms is the series transform of convolution.

We also obtain here the inverse transform of (1.3) which has already been derived by the authors in Singhal and Savita Kumari (1982) by a method entirely different from the one used here.

2. Proof of the Theorem

We first observe that

$$\sum_{n=0}^{\infty} F(n)\, t^n = (1 + yt^m)^p \sum_{n=0}^{\infty} f(n)\, z^n \qquad \ldots(2.1)$$

and

$$\sum_{n=0}^{\infty} G(n)\, t^n = (1 + yt^m)^q \sum_{n=0}^{\infty} g(n)\, z^n \qquad \ldots(2.2)$$

where

$$z = \frac{t}{(1 + yt^m)^\lambda} \cdot \qquad \ldots(2.3)$$

Now it is easy to see that

$$\sum_{n=0}^{\infty} (F * G)(n)\, t^n = \sum_{n=0}^{\infty} G(n)\, t^n \sum_{j=0}^{\infty} F(j)\, t^j$$

$$= (1 + yt^m)^{p+q} \sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} f(j)\, g(n-j)$$

$$= (1 + yt^m)^{p+q} \sum_{n=0}^{\infty} z^n (f * g)(n)$$

$$= \sum_{n=0}^{\infty} S_{p+q} [(f * g)(n)] t^n$$

whence it follows that

$$(F * G)(n) = S_{p+q} [(f * g)(n)].$$

If the various parameters involved in the series transform (1.3) are particularized suitably, so as to yield the explicit representation of the generalized Humbert polynomials $P_n(m, x, y, p, C)$ introduced by Gould (1965), the relation (1.9) would simplify to

$$\sum_{j=0}^{n} P_j(m, x, y, p, C)\, P_{n-j}(m, x, y, q, C) = P_n(m, x, y, p + q, C) \quad \ldots (2.4)$$

whereas the simplified form of (1.9), in the case of Laguerre polynomials, is the well known relation [Rainville (1960, p. 209, eqn. 3)].

$$\sum_{j=0}^{n} L_j^{(\alpha)}(x)\, L_{n-j}^{(\beta)}(x) = L_n^{1+\alpha+\beta}(2x). \quad \ldots (2.5)$$

3. Inverse Transform

In order to obtain the inverse transform of (1.3), we begin with the relation (2.1) namely,

$$\sum_{n=0}^{\infty} f(n) z^n = (1 + yt^m)^{-p} \sum_{n=0}^{\infty} F(n) t^n$$

$$= \sum_{n=0}^{\infty} F(n) z^n (1 + yt^m)^{\lambda n - p} \quad \ldots (3.1)$$

which in view of the well known Lagrange Expansion (Pólya and Szegö 1972, p. 348, Problem 212)

$$x^a = \sum_{k=0}^{\infty} \frac{a}{a + bk} \binom{a + bk}{k} u^k,\ u = (x - 1) x^b$$

takes the form

$$\sum_{n=0}^{\infty} f(n) z^n = \sum_{n=0}^{\infty} F(n) z^n$$

$$\times \sum_{k=0}^{\infty} \frac{\lambda n - p}{\lambda n - p + \lambda mk} \binom{\lambda n - p + \lambda mk}{k} (yz^m)^k \quad \ldots (3.2)$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{[n/m]} F(n-mk) \frac{\lambda n-\lambda mk-p}{\lambda n-p} \binom{\lambda n-p}{k} y^k z^n.$$

Thus it follows that

$$f(n) = \sum_{k=0}^{[n/m]} y^k \frac{p-\lambda n+\lambda mk}{p-\lambda n} \binom{\lambda n-p}{k} F(n-mk). \quad \ldots (3.3)$$

The relation (3.3) may also be stated in the form

$$(f * g)(n) = \sum_{k=0}^{[n/m]} y^k \frac{p+q-\lambda n+\lambda mk}{p+q-\lambda n} \binom{\lambda n-p-q}{k} \times (F * G)(n-mk). \quad \ldots (3.4)$$

It is also worth mentioning here that if we start with (3.3) and do analogous analysis we shall get (1.3) as the inverse of (3.3). Thus the technique of this section provides us an alternative proof of the theorem given by the authors [Singhal and Savita Kumari 1982].

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