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A NOTE ON CONVOLUTION IN BANACH SPACES

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A convolution operation is defined on Banach spaces and some of its properties are investigated. It is proved that under a density condition, the reflexivity of a Banach space implies the reflexivity of the space of operators on it.

The following very general form of the convolution operation between linear functionals was formulated in Rao Chivukula and Heckman (1974) and there we showed how it generalizes the definition of Hewitt and Ross (1963, p. 262), which itself is a generalization of the classical notions of convolutions of functions and measures (see section 1 in Rao Chivukula and Heckman 1974). We used the definition to prove several factorization theorems. In this note we show how this definition can be applied to formulate a convolution operation in Banach spaces; a very brief mention of this has been made in Rao Chivukula and Heckman (1974, p. 149).

We apply this convolution operation to show (in Theorem 2, below) that when a certain density condition is satisfied, the reflexivity of a Banach space X imples the reflexivity of $B(X, X)$, the space of bounded linear operators on X The converse is always true; see Remark 1, below. Of course, if X is finite demensional, then $B(X, X)$ is also finite dimensional and hence reflexive. If X is infinite dimensional then it is not known, in general, if the reflexivity of X implies that of $B(X, X)$. However, it is well known that $B(H, H)$ is not reflexive where H is an infinite dimensional Hilbert space; see Remark 2, below. It has been proved (Kalton 1974) that $B(X, X)$ is not reflexive if X is reflexive and nonseparable; and it has also been shown (Holub 1973) that if X is reflexive and has the approximation property, then $B(X, X)$

is not reflexive. Since there exist (separable) Banach spaces without the approximation property (Enflo 1973), our result could be applied to them; see Remark 3, below.

Definition—Let S and T be sets and let $\mathscr{F}_S$ and $\mathscr{F}_T$ be linear spaces of real or complex valued functions on S and T respectively. Suppose in addition that S is a collection of transformations of T For $f \in \mathscr{F}_T$ and $s \in S$, define $(sf)(t) = f(s(t))$ for all $t \in T$; and suppose that for each $f \in \mathscr{F}_T$ and $s \in S$, $sf \in \mathscr{F}_T$ (in other words, $\mathscr{F}_T$ is S-invariant). If M is a linear functional on $\mathscr{F}_T$ and if $f \in \mathscr{F}_T$, we denote by $\bar{M}f$ the function on S defined as $(\bar{M}f)(s) = M(sf)$ for all $s \in S$. Suppose further that M is such that $\bar{M}f \in \mathscr{F}_S$ for all $f \in \mathscr{F}_T$. Now if L is any linear functional on $\mathscr{F}_S$, then the function whose value at any $f \in \mathscr{F}_T$ is $L(\bar{M}f) = (L \circ \bar{M})(f)$ is well defined. We call this function the convolution of the functionals L and M and denote it by $L * M$. (It is easy to verify that $L * M$ is a linear functional on $\mathscr{F}_T$).

For a Banach space X, we write X' and X'' for the first and second normed conjugate spaces (respectively) and $B(X, X)$ for the Banach algebra of all bounded linear operators from X into X. The conjugate spaces $B(X, X)'$, $B(X, X)''$ and the space of operators $B(X'', X'')$ are similarly defined.

To apply the above definition of convolution to Banach spaces, let $T = X$ a Banach space and $S = B(X, X)$ the space of operators; further let $\mathscr{F}_S = B(X, X)'$ and $\mathscr{F}_T = X'$. Now the following four facts are easily verified: (i) For every $F \in X'$ and $g \in B(X, X)$, ${}_gF \in X'$; (ii) For every $M \in X'$, and $F \in X'$, $\bar{M}F \in B(X, X)'$ where $(\bar{M}F)(g) = M({}_gF)$ for all $F \in X'$. Hence if $L \in B(X, X)''$, then $(L * M)(F) = L(\bar{M}F)$ is well defined for all $F \in X'$ and $M \in X''$. Moreover, (iii) for every $M \in X''$ and $L \in B(X, X)''$ $L * M \in X''$. Thus we have $B(X, X)'' * X'' \subseteq X''$ and each $L \in B(X, X)''$ generates a convolution operator $\hat{L}$, mapping X'' into X'' defined by the equation

$$\hat{L}(M) = L * M.$$

It is also easy to verify that (iv) For eaoh $L \in B(X, X)''$, $\hat{L}$ is a bounded linear operator on X''; that is, $\hat{L} \in B(X'', X'')$.

Let (as usual), Q denote the natural embedding of $B(X, X)$ into its second conjugate space $B(X, X)''$; that is $(Qg)(G) = G(g)$ for all $g \in B(X, X)$ and $G \in B(X, X)'$. It is well known that Q is a linear isometry mapping $B(X, X)$ into $B(X, X)''$. Next for each $g \in B(X, X)$, $M \in X''$ and $F \in X'$, let

$$(g^{**}M)(F) = M({}_gF).$$

Since by definition, $M({}_gF) = (\bar{M}F)(g) = (Qg)(\bar{M}F)$, this shows that $g^{**}M \in X''$ and that g^{**} is a linear mapping of X'' into itself. Moreover, since $Qg \in B(X, X)''$, we can consider $\hat{Qg}$ as defined in (iv); and the above equations combined with the fact that $((Qg) * M)(F) = (Qg)(\bar{M}F)$, show that

$$Qg = g^{**};$$

and it follows directly from definitions that

$$\|g\| = \|Qg\| = \|g^{**}\|.$$

Thus $g^{**} \in B(X'', X'')$; and since each of the operators g, Qg and g^{**} is linear, the above equation shows that the mapping $Qg \to g^{**}$ is one-one. The following theorem collects the main properties of this convolution operation. But first, if W is a subset of a linear space V, SpW denotes linear span of W.

Theorem 1—(a) Suppose $Sp\{\bar{M}F \mid M \in X'', F \in X'\}$ is dense in $B(X, X)'$. If $L \in B(X, X)''$, then $\hat{L} = g^{**}$ for some $g \in B(X, X)$ if and only if $L = Qg$. (b) The mapping $L \to \hat{L}$ is linear and continuous from $B(X, X)''$ into $B(X'', X'')$. In fact $\|\hat{L}\| \leqslant \|L\|$. (c) The mapping $L \to \hat{L}$ is one-one if and only if $Sp\ \{\bar{M}F \mid M \in X''$, $F \in X'\}$ is dense in $B(X, X)'$.

PROOF : (a) We have already seen above, that if $L = Qg$ then $\hat{L} = \hat{Qg} = g^{**}$ So we only need prove the converse. Suppose $\hat{L} = g^{**}$ for some $g \in B(X, X)$. Then for any $M \in X''$ and $F \in X'$,

$$L(\bar{M}F) = (L * M)(F) = (\hat{L}(M))(F) = (g^{**}(M))(F) = M({}_gF)$$
$$= (\bar{M}F)(g) = (Qg)(\bar{M}F)$$

which shows that $L = Qg$ on $Sp\{\bar{M}F\}$, but since this latter set is dense in $B(X, X)'$ it follows that $L = Qg$.

(b) The linearity is easily checked. For $M \in X''$.

$$\|\hat{L}(M)\| = \|L * M\| = \sup_{\substack{\|F\| = 1 \\ F \in X'}} |(L * M)(F)|$$

$$= \sup_{\|F\| = 1} |L(\bar{M}F)| \leqslant \sup_{\|F\| = 1} \|L\| \cdot \|\bar{M}F\|$$

$$\leqslant \sup_{\|F\| = 1} \|L\| \cdot \|M\| \cdot \|F\| = \|L\| \cdot \|M\|.$$

Now by taking sup over $\|M\| = 1$, we have $\|\hat{L}\| \leqslant \|L\|$.

(c) Since the operator $\hat{L}$ acts only on the closure of $Sp\{\bar{M}F\}$, the density of the latter space implies that the mapping $L \to \hat{L}$ is one-one.

Conversely suppose that the mapping $L \to \hat{L}$ is one-one. Let $L \in B(X, X)''$ and define H on the set $\{\bar{M}F \mid M \in X'', F \in X'\}$ by $H(\bar{M}F) = (\hat{L}(M))(F)$. Then H is well defined and can be linearly extended to $Sp\{\bar{M}F\}$, where it coincides with L. But then by Hahn-Banach theorem, we can extend H continuously to all of $B(X, X)'$. But Since $H = L$ on $Sp\{\bar{M}F\}$, it follows that $\hat{H} = \hat{L}$. This means that each L in $B(X, X)''$ is completely determined by its values on $Sp\{\bar{M}F\}$ which cannot happen unless $Sp\{MF\}$ is dense in $B(X, X)'$; and the theorem is completely proved.

Before stating the next theorem we recall the three mappings defined above:

(1) The mapping $Q : B(X, X) \to B(X, X)'' : g \to Qg$ defined by $(Qg)(G) = G(g)$ for all $G \in B(X, X)'$.

(2) The mapping** : $B(X, X) \to B(X'', X'') : g \to g^{**}$.

(3) The mapping $\hat{}$: $B(X, X)'' \to B(X'', X'') : L \to \hat{L}$.

We have shown that $\widehat{Qg} = g^{**}$; and the mappings Q and "**" are linear, one-one and isometric: and "^" is also linear; and if $Sp\{\bar{M}F\}$ is dense in $B(X, X)'$, then it is one-one and norm decreasing. In view of these relations, we identify each g in $B(X, X)$ with Qg and each Qg with $\widehat{Qg} = g^{**}$. This means that $B(X, X)$ may be identified with (i.e., can be considered linearly isometric to) a subspace of $B(X'', X'')$. Also we have seen that each L in $B(X, X)''$ can be considered as an operator $\hat{L}$ in $B(X'', X'')$ and if $Sp\{\bar{M}F\}$ is dense $B(X, X)'$, then the mapping $L \to \hat{L}$ is one-one. Thus we write symbolically

$$B(X, X) \subset B(X, X)'' \subset B(X'', X'')$$

where the first inclusion is a linear isometry (namely, the map Q) and the second inclusion is one-one linear and continuous (the map ^). Also $B(X, X)$ may be considered directly as a subspace of $B(X'', X'')$ via the map $g \to g^{**}$. Now if X is a reflexive Banach space then $B(X'', X'')$ can be regarded as the same as $B(X, X)$ and the above chain of containment relations indicates the $B(X, X)$ and its second conjugate $B(X, X)''$ coincide and that $B(X, X)$ is reflexive. Our next theorem shows that this is so.

Theorem 2—Suppose X is a reflexive Banach space and $Sp\{\bar{M}F \mid M \in X'', F \in X'\}$ is dense is $B(X, X)'$. Then $B(X, X)$ is reflexive and hence $B(X, X)''$ is isometrically isomorphic to $B(X'', X'')$.

PROOF: The density of $Sp\{\bar{M}F\}$ implies that the mapping $L \rightarrow \hat{L}$ is one-one. In this proof we let Q denote both of the cannonical isometries of $B(X, X)$ into $B(X, X)''$ and of X into X''. This will cause no confusion since from the context it will be clear which mapping is being considered. Now X is reflexive means Q is onto; and recall that for each $g \in B(X, X)$, $\hat{Q}g = g^{**}$. First we show that the mapping "$**$" is onto; (we already know that it is linear, isometric and one-one).

For each $H \in B(X'', X'')$, if $g = Q^{-1}HQ$, then g is an element of $B(X, X)$ where Q (and Q^{-1}) is the canonical isometry (and its inverse) of X onto X''. Now let $M \in X''$ and take $x \in X$ such that $M = Qx$; and let $y = (Q^{-1}HQ)(x)$, that is, $Qy = HQx$. For any $F \in X'$,

$$\begin{aligned}(g^{**}M)(F) &= (g^{**}Qx)(F) = (Qx)({}_gF) = ({}_gF)(x) \\ &= F(gx) = F(Q^{-1}HQx) = F(y) = (Qy)(F) \\ &= (HQx)(F) = (HM)\ (F)\end{aligned}$$

which shows that $g^{**} = H$ and so the mapping "$**$" is onto. Now in the equation $\wedge \circ Q = **$, the mapping $Q : B(X, X) \rightarrow B(X, X)''$ is linear and isometric and the mapping "$\wedge$" is one-one. We have just proved that "$**$" is onto. These facts imply that the canonical map Q is onto; that is $B(X, X)$ is reflexive and the theorem is proved.

Remarks: (1) One can prove by using well known characterizations of reflexivity of Banach spaces (for example, that the unit ball is weakly compact) that the reflexivity of the operator space $B(X, X)$ implies the reflexivity of X. A detailed proof is given in Heckman (1971). Our Theorem 2, above, shows that the converse holds under the additional assumption $Sp\{\bar{M}F\}$ is dense in $B(X, X)'$.

(2) That $B(H, H)$ is not reflexive for an infinite dimensional Hilbert space H can be seen as follows. Let $\mathscr{T} = \mathscr{T}(H)$ and $\mathscr{K} = \mathscr{K}(H)$ denote respectively the trace class and compact operators in $B(H) = B(H, H)$. As is well known (via the duality induced by the trace functional), $\mathscr{K}' \cong \mathscr{T}$ and $\mathscr{T}' \cong B(H)$. Thus $B(H) \cong \mathscr{K}''$ is not reflexive unless $B(H) \cong \mathscr{K}$, that is, unless H is finite dimensional.

(3) As noted earlier in the introduction, our Theorem 2 has significance only for separable reflexive Banach spaces without the approximation property. It would be interesting to know if any of them satisfy our density condition.

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