Indian J. pure appl. Math., 14(12) : 1534–1541, December 1983

UNSTEADY AXISYMMETRIC ROTATIONAL FLOW OF ELASTICO-VISCOUS LIQUID

SWAPNA MUKHERJEE

Department of Mathematics, Indian Institute of Technology, Kharagpur

AND

SRIKUMAR MUKHERJEE

Department of Mathematics, Malda College. Malda, West Bengal

(*Received* 25 *January* 1983)

The rotational flow of an elastico-viscous liquid due to the time-dependent rotation of a circular cylinder is studied in view of its growing importance in various technical problems. The exact velocity profile of elastico-viscous liquid and the skin-friction are obtained by Laplace transform technique. The effects of elastic elements on velocity and skin-friction at the boundary have been studied.

1. INTRODUCTION

A numer of workers have studied both steady and unsteady two-dimensional axisymmetric rotational flow of a viscous fluid due to its varied and wide applications in the problems of technological interest. Khamrui (1957) has analysed the slow steady motion of an infinite viscous fluid due to the rotation of a circular cylinder. Iben (1974) has considered the non-stationary, plane and circular-symmetric flow of a viscous fluid which forms itself within as well as outside a rotating and infinitely long cylinder and obtained a complete analytical solution for any boundary and initial conditions using the method of Laplace transforms. Subsequently Reismann (1975) has developed a solution of the problem of two-dimensional axisymmetric rotational flow of a viscous fluid annulus bounded by two concentric circles considering a suitable modified eigenfunction approach. Recently Mukherjee and Bhattacharya (1982) have studied the rotational flow of viscous liquid due to the rotation of a circular lamina or by the action of shearing stress along the boundary.

The purpose of the present paper is to extend the analysis of Mukherjee and Bhattacharya (1982) to cover a wider class of liquid, namely elastico-viscous liquid, and, in particular to observe the effects of elastic elements in the liquid. The steady and the transient components of the velocity field are obtained explicitly. The transient effects are shown to die out exponentially in the limit $t \to \infty$ and the ultimate steady flow is established. It is observed that both steady state and transient solutions are independently modified by the elastic elements in the liquid. The effects of elastic elements on the velocity profile and skin-friction are studied graphically. It is found that the phase of oscillation of the velocity profile is quicker in elastico-viscous liquid, and the elastic element increases the skin-friction at any point on the boundary of the circular cylinder.

2. Formulation of the Problem

Our investigation is based on a model of elastico-viscous liquids which retain essentially the rheological properties of such liquids under common operating conditions. The constitutive equation of such incompressible liquids, introduced by Oldroyd (1950), is given by

$$p'_{ik} + \lambda_1 \frac{D}{Dt} p'_{ik} - \lambda_1 (p'_{ij} e_{ik} + p'_{jk} e_{ij})$$

$$= 2\eta_0 \left[e_{ik} + \lambda_2 \frac{D}{Dt} e_{ik} - 2\lambda_2 e_{ij} e_{jk} \right] \qquad \ldots(1)$$

where $p_{ik} = - p \delta_{ik} + p'_{ik}$ the stress-tensor, $e_{ik} = \frac{1}{2} (v_{i,k} + v_{k,i})$ the rate of strain-tensor, p an arbitrary isotropic pressure, δ_{ik} the metric tensor of a fixed coordinate x_i, η_0 a coefficient of viscosity, λ_1 a stress-relaxation time, $\lambda_2 (< \lambda_1)$ a rate of strain retardation time and the operator D/Dt denotes the convective time derivative.

We consider the axisymmetric flow of an elastico-viscous liquid obeying (1) within a circular cylinder of radius a. The time-dependent excitation is caused by the prescribed time-dependent angular velocity to the circular cylinder (Fig. 1). It is assumed that the liquid responds in circular motion about the centre and hence the motion of a liquid particle is a function of radial coordinate r and time t only.

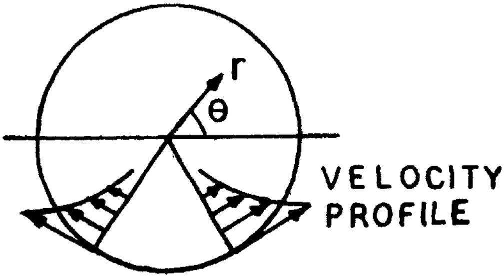

Fig. 1. Sketch of the physical problem (internal flow)

The equations governing the elastico-viscous liquid flow are

$$\left(1+\lambda_1 \frac{\partial}{\partial t}\right) \frac{\partial v}{\partial t} = \nu \left(1+\lambda_2 \frac{\partial}{\partial t}\right) \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2}\right) \qquad \ldots(2)$$

$$\rho \frac{v^2}{r} = \frac{\partial p}{\partial r} \qquad \ldots(3)$$

where $v(r, t)$ is the velocity component of the liquid in the direction of increasing θ and ν is the kinematic coefficient of viscosity.

The admissible boundary conditions require the specification of time-dependent velocity on the external boundary and at the centre of the circular cylinder given by $t > 0$:

$$\left.\begin{array}{ll} v = v_0\, e^{-i\Omega t} & \text{on } r = a \\ v \text{ is finite} & \text{on } r = 0 \end{array}\right\} \qquad \ldots(4a)$$

where v_0 is the representative velocity and Ω is the imposed oscillation. For a liquid at rest for all $t < 0$ it may be assumed that the initial state of stress is zero. The initial conditions are

$$v(r, t) = \frac{\partial v(r, t)}{\partial t} = 0 \quad \text{at } t = 0 \quad \text{and for all } r. \qquad \ldots(4b)$$

Introducing the following dimensionless quantities

$$\bar{v} = \frac{v}{v_0}, \; \bar{r} = \frac{r}{a}, \; \bar{t} = \frac{t\nu}{a^2}, \; \alpha_1 = \lambda_1 \frac{\nu}{a^2}$$

$$\alpha_2 = \lambda_2 \frac{\nu}{a^2}, \; \bar{\Omega} = \Omega \frac{a^2}{\nu}, \; \bar{p} = \frac{pa^2}{\rho\nu^2}$$

eqns. (2) and (3) become (dropping bar)

$$\left(1+\alpha_1 \frac{\partial}{\partial t}\right) \frac{\partial v}{\partial t} = \left(1+\alpha_2 \frac{\partial}{\partial t}\right) \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2}\right) \qquad \ldots(5)$$

$$\frac{v^2}{r} = \frac{\partial p}{\partial r} \qquad \ldots(6)$$

subject to the boundary and initial condition

$t > 0$:

$$\left.\begin{array}{ll} v = e^{-i\Omega t} & \text{on } r = 1 \\ v \text{ is finite} & \text{on } r = 0 \end{array}\right\} \qquad \ldots(7a)$$

and

$$v = \frac{\partial v}{\partial t} = 0 \quad \text{at } t = 0. \qquad \ldots(7b)$$

3. Solution of the Problem

The present problem is solved by using the technique of integral transform. We apply Laplace transform to eqns. (5) and (7a) subject to the condition (7b), we get

$$\frac{d^2 \bar{v}}{dr^2} + \frac{1}{r}\frac{d\bar{v}}{dr} - \left[\frac{p(1+\alpha_1 p)}{(1+\alpha_2 p)} + \frac{1}{r^2}\right]\bar{v} = 0 \qquad \ldots(8)$$

$$\left.\begin{aligned} \bar{v} &= \frac{1}{p+i\Omega} \quad \text{on } r = 1 \\ \bar{v} &\text{ is finite} \quad \text{on } r = 0 \end{aligned}\right\} \qquad \ldots(9)$$

where

$$\bar{v} = \int_0^\infty v\, e^{-pt}\, dt, \quad \mathrm{Re}\,(p) > 0. \qquad \ldots(10)$$

The solution of eqn. (8) subject to the boundary conditions in (9) can be written as

$$\bar{v} = A\,(p)\, I_1\left(r\sqrt{\frac{p\,(1+\alpha_1 p)}{(1+\alpha_2 p)}}\right) \qquad \ldots(11)$$

where

$$A\,(p) = \frac{1}{(p+i\Omega)\, I_1\left(\sqrt{\frac{p\,(1+\alpha_1 p)}{(1+\alpha_2 p)}}\right)}. \qquad \ldots(12)$$

The inverse Laplace tranformation corresponding to (11) results the velocity profile as

$$v = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{I_1\left(r\sqrt{\frac{p\,(1+\alpha_1 p)}{(1+\alpha_2 p)}}\right)}{(p+i\Omega)\; I_1\left(\sqrt{\frac{p\,(1+\alpha_1 p)}{(1+\alpha_2 p)}}\right)}\, e^{pt}\, dp \qquad \ldots(13)$$

where γ is greater than the real part of the singularities of the integrand. Evaluation of the above integral requires a knowledge of the nature and location of the singularities of the integrand. It is clear that the integrand is an integral function of p and has only simple pole at $p = -i\Omega$ and poles at the zeros ($p = p_n$) of

$$I_1\left(\sqrt{\frac{p\,(1+\alpha_1 p)}{(1+\alpha_2 p)}}\right) \equiv Q\,(p) \qquad \text{(say).}$$

To find the zeros of the above expression, we put

$$\frac{p\,(1+\alpha_1 p)}{(1+\alpha_2 p)} = -\beta^2 \qquad \ldots(14)$$

in the expression and it becomes $i\, J_1\,(\beta)$, where β_n's are the roots of

$$J_1\,(\beta) = 0. \qquad \ldots(15)$$

The zeros of $Q(p)$ is then

$$p\,(=p_n) = \frac{1}{2\alpha_1}\left[-(1+\alpha_2\beta_n^2) \pm \sqrt{(1+\alpha_2\beta_n^2)^2 - 4\alpha_1\beta_n^2}\right]. \quad \ldots(16)$$

Poles p_n $(n = 1, 2, 3 \ldots)$ are all simple poles when $(1+\alpha_2\beta_n^2)^2 \gtrless 4\alpha_1\beta_n^2$ since for the first inequality the poles are all negative real numbers and for the second one the poles are complex conjugates and all lie within the circle $|p_n| = \beta_n/\sqrt{\alpha_1}$. It may be remarked that the present problem results in a set of double poles $p_n = -\beta_n/\sqrt{\alpha_1}$ when $(1+\alpha_2\beta_n^2)^2 = 4\alpha_1\beta_n^2$ which is found absent in the case of Newtonian fluid (Mukherjee *et al.* 1982).

To evaluate the integral in (13) we first consider the case when all the singularities are simple poles. The residue at the simple pole $p = -i\Omega$ is

$$\frac{I_1\left(r\sqrt{\dfrac{-i\Omega(1-i\Omega\alpha_1)}{(1-i\Omega\alpha_2)}}\right)}{I_1\left(\sqrt{\dfrac{-i\Omega(1-i\Omega\alpha_1)}{(1-i\Omega\alpha_2)}}\right)}\, e^{-i\Omega t}.$$

The residue at the simple pole $p = p_n$ is

$$-\frac{2\beta_n}{R_n}\,\frac{J_1(r\beta_n)}{J_1'(\beta_n)}\,\frac{(\alpha_2 p_n + 1)}{(p_n + i\Omega)}\, e^{p_n t}$$

where

$$R_n = \pm\,[(1+\alpha_2\beta_n^2)^2 - 4\alpha_1\beta_n^2]^{1/2}.$$

The expression for the velocity profile can be written as

$$v = \frac{I_1\left(r\sqrt{\dfrac{-i\Omega(1-i\Omega\alpha_1)}{(1-i\Omega\alpha_2)}}\right)}{I_1\left(\sqrt{\dfrac{-i\Omega(1-i\Omega\alpha_1)}{(1-i\Omega\alpha_2)}}\right)}\, e^{-i\Omega t}$$

$$-\,2\sum_n \frac{\beta_n}{R_n}\,\frac{(\alpha_2 p_n + 1)}{(p_n + i\Omega)}\,\frac{J_1(r\beta_n)}{J_1'(\beta_n)}\, e^{p_n t} \quad \ldots(17)$$

$$= v_{St} + v_{Tt}$$

where v_{St} represents the steady state solution which is essentially made up of the residue contribution from the pole at $p = -i\Omega$ and v_{Tt} represents the transient part arising out of the residues at $p = -p_n$. If we put $\alpha_1 = \alpha_2 = 0$ in (17), the velocity profile thus obtained is in good agreement with Mukherjee *et al.* (1982) for Newtonian fluid.

For the case of double pole occurring at $p_n = -\beta_n/\sqrt{\alpha_1}$ $(n = 1, 2, \ldots)$, the velocity distribution is given by

$$v = v_{St} + \sum_n M \exp(-\beta_n t/\sqrt{\alpha_1}) + \sum_n t\, N \exp(-\beta_n t/\sqrt{\alpha_1}) \qquad \ldots(18)$$

$$= v_{St} + v'_{Tt} + v''_{Tt}$$

where

$$M = \left[\frac{d}{dp} f(p) \left\{p + \beta_n/\sqrt{\alpha_1}\right\}^2\right]_{p = -\beta_n/\sqrt{\alpha_1}}$$

$$N = \left[(p + \beta_n/\sqrt{\alpha_1}) f(p)\right]_{p = -\beta_n/\sqrt{\alpha_1}}$$

$$f(p) = \frac{I_1(rm)}{I_1(m)(p + i\Omega)} \quad \text{and} \quad m = \frac{p(1 + \alpha_1 p)}{(1 + \alpha_2 p)}.$$

The non-dimensional skin-friction on the wall of the circular cylinder is given by

$$\tau_{r\theta}\Big|_{r=1} = \frac{a\, p'_{r\theta}}{\eta_0 v_0}\Big|_{r=1}$$

$$= \left[1 - (\alpha_1 - \alpha_2)\frac{\partial}{\partial t}\right]\left[\frac{\partial v}{\partial r} - \frac{v}{r}\right]_{r=1}$$

$$= \frac{1}{I_1(K_1)}\left[1 + (\alpha_1 - \alpha_2) i\Omega\right] e^{-i\Omega t}\left[K_1 I_1'(K_1) - I_1(K_1)\right]$$

$$- \sum_n \left[1 - (\alpha_1 - \alpha_2) p_n\right] K_n e^{p_n t} \left[\beta_n J_1'(\beta_n)\right] \qquad \ldots(19)$$

where

$$K_1 = \sqrt{\frac{-i\Omega(1 - i\Omega\alpha_1)}{(1 - i\Omega\alpha_2)}} \quad \text{and} \quad K_n = \frac{2\beta_n(\alpha_2 p_n + 1)}{R_n J_1'(\beta_n)(p_n + i\Omega)}.$$

4. Discussions

A notable feature of (17) and (18) is that the effect of elastic elements in the liquid is reflected on both the steady and transient solution. With certain exception, the characteristic features of the velocity field are somewhat similar to those of the time-dependent rotation of the circular cylinder in a viscous fluid. v_{St} represents the oscillatory part whereas v_{Tt} in (17) and v'_{Tt} in (18) represent the transient parts. As $t \to \infty$, the transient parts die out exponentially and the motion finally attains the oscillatory motion about the steady state. However, the steady state solution can also be derived directly from eqn. (8). It may be noted from eqn. (17) that, if we neglect the terms of $O(r^2)$, the velocity profile of the elastico-viscous liquid has a boundary layer character on the circular cylinder.

The velocity profile of elastico-viscous liquid for different values of elastic elements α_1 and α_2 ($< \alpha_1$) and also viscous liquid are plotted against time t in Fig 2. From the figure, it is evident that the effect of elasticity is to diminish the phase of oscillation of the velocity profile i.e., the rate of change of phase of oscillation is quicker in elastico-viscous liquid and the amplitude changes in a parabolic way as in the case of ordinary viscous flow. The effect of elasticity on skin-friction at the boundary of the circular cylinder is also shown in Fig. 3. It clearly reveals that elastic element increases the skin firiction at any point on the boundary of circular cylinder.

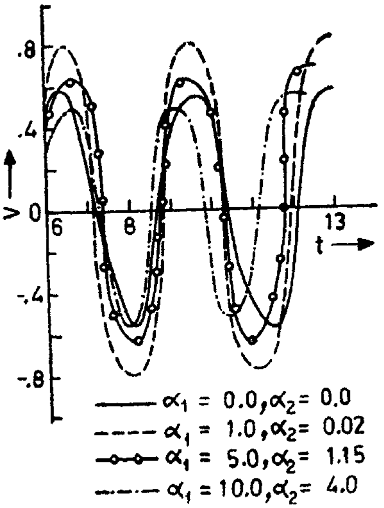

FIG. 2. Velocity profile of elastico-viscous and viscous liquid when $\Omega = 2$.

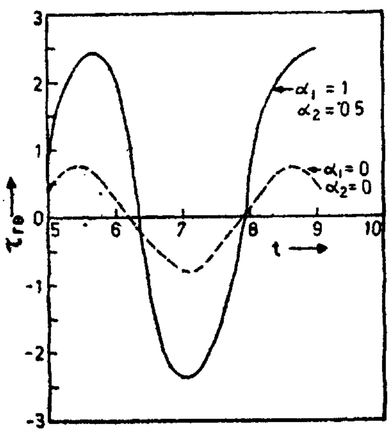

FIG. 3. Effect of elastic elements on shearing stress when $\Omega = 2$.

ACKNOWLEDGEMENT

The authors are greatful to Dr M. K. Maiti for his constant encouragement and guidance throughout the preparation of this paper.

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