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A NEW MULTIVARIATE PROBABILITY DENSITY FUNCTION

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A new multivariate probability density function of the implicit type is introduced in this paper. Its characteristic function is also obtained. The inversion theorem for the characteristic function (showing the uniqueness phenomenon between the probability density function and the characteristic function) is also stated. Finally, an example showing the reduction of density function and also of its characteristic function to a bivariate case is given in the last section. The reduction to the well known exponential probability law is O also shown.

§1. We present in this paper a new theoretical joint probability density function (pdf) for the set of r independent random variables $(X_1,...,X_r)$. We also establish the characteristic function for this set of random variables.

The special features of this pdf are (i) it is a function of r $(r = 1,2,...)$ and hence actually represents a family of pdf's and (ii) it is an implicit function so that separation into the individual densities for X_1, X_2 ... is not possible. The well known bivariate normal pdf (Feller 1969, p. 69) centred at the origin viz :

$$f(t_1, t_2) = \frac{1}{2\pi\,\sigma_1\,\sigma_2\,\sqrt{1-\rho^2}} \times \exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{t_1^2}{\sigma_1^2} - 2\rho\,\frac{t_1\,t_2}{\sigma_1\,\sigma_2} + \frac{t_2^2}{\sigma_2^2}\right)\right],$$

$$\sigma_1, \sigma_2 > 0\ ;\ -1 < \rho < 1$$

is an example of a pdf in which a separation into individual densities is not possible.

The multivariate pdf presented here may be used as a model for certain kinds of random phenomena like the lifetime of objects which is affected by several parameters.

§2. It is known that if $X_1, \ldots, X_r$ is a set of r independent random vriables (r. v.) its joint cummulative distribution function (cdf) is defined as (Papoulis 1965, Ch. 8) :

$$F(t_1,\ldots, t_r) = P\{X_1 \leqslant t_1,\ldots, X_r \leqslant t_r\}$$

and that its joint pdf is given by

$$f(t_1,\ldots,t_r) = \partial^r F(t_1,\ldots, t_r) / \partial t_1 \ldots \partial t_r.$$

The function $f(.)$ so defined is everywhere non-negative and its integration over the entire range of variation of $t_1,\ldots,t_r$ equals unity i. e.,

$$\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} f(t_1, \ldots,t_r)\, dt_1 \ldots dt_r = 1. \qquad \ldots(2.1)$$

Also, the characteristic function (c. f.) of a random variable has a two-fold importance : (i) it is an additional mathematical tool for studying the mathematical behaviour of the r. v. and (ii) since the c. f. and its inversion are unique the r. v. might be specified either by its pdf or by its c. f.

The characteristic function of the set of r. v. $(X_1,\ldots, X_r)$ which is governed by the pdf $f(t_1,\ldots,t_r)$ is defined as (Papoulis 1965, p. 244) :

$$\phi(w_1,\ldots, w_r) = E\{e^{(iw_1X_1+\cdots+w_rX_r)}\}$$

$$= \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} e^{i(w_1t_1+\cdots+w_rt_r)} f(t_1,\ldots, t_r)\, dt_1 \ldots dt_r \qquad \ldots(2.2)$$

in which $E\{.\}$ represents the expectation and $i = \sqrt{-1}$. We now state without proof the inversion theorem of the c. f.

Inversion Theorm

If $\phi\ (w_1,\ldots, w_r)$ is the characteristic function of the set of r. v. $(X_1,\ldots,X_r)$ with pdf $f(t_1,\ldots,t_r)$, as defined by (2.2), then

$$\frac{1}{(2\pi)^r} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} e^{-i(t_1w_1+\cdots+t_rw_r)}\, \phi\,(w_1,\ldots,w_r)\, dw_1 \ldots dw_r$$

$$= f(t_1,\ldots, t_r). \qquad \ldots(2.3)$$

The theorem can be proved by using the following two properties of the Dirac delta function $\delta(t)$:

$$\text{(i)} \int_{-\infty}^{\infty} f(y)\, \delta(x - y)\, dy = \int_{-\infty}^{\infty} f(y)\, \delta(y - x)\, dy = f(x)$$

and

$$\text{(ii)} \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i(x-a)w}\, dw = \delta(x-a).$$

§3. For introducing the new pdf we first establish the corresponding multiple (r-ple) integral involving Fox's H-function as modified by Braaksma (1963) viz.,

$$H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} \{(a_p, A_p)\} \\ \{(b_q, B_q)\} \end{matrix} \right]$$

$$= \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j - B_j\xi) \prod_{j=1}^{n} \Gamma(1 - a_j + A_j\xi)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + B_j\xi) \prod_{j=n+1}^{p} \Gamma(a_j - A_j\xi)} z^{\xi}\, d\xi$$

with $1 \leqslant m \leqslant q$, $0 \leqslant n \leqslant p$; $z \neq 0$; A_j $(j = 1, \ldots p)$ and B_j $(j = 1, \ldots, q)$ are positive numbers, a_j $(j = 1, \ldots, p)$ and b_j $(j = 1, \ldots, q)$ are complex numbers such that $A_j(b_h + \nu) \neq B_h(a_j - 1 - \lambda)$ for $\nu, \lambda = 0, 1, \ldots$; $h = 1, \ldots, m$ and $j = 1, \ldots, n$. The contour L and the conditions for the existence of the H-function have been fully discussed by Braaksma.

The G-function is a particular case of the H-function with all $A_j, B_j = 1$.

By direct evaluation we get

$$I_1 = \int_0^{\infty} t_1^{\rho_1 - 1} \exp(-t_1)\, H_{p,q}^{m,n}\left[\alpha t_1 \,\middle|\, \begin{matrix} \{(a_p, A_p)\} \\ \{(b_q, B_q)\} \end{matrix} \right] dt_1$$

$$= H_{p+1,q}^{m,n+1}\left[\alpha \,\middle|\, \begin{matrix} (1-\rho_1, 1), \{(a_p, A_p)\} \\ \{(b_q, B_q)\} \end{matrix} \right] \qquad \ldots(3.1)$$

provided $\rho_1 + \min \operatorname{Re}(b_h/B_h) > 0$, $(h = 1, \ldots, m)$ and

$$\delta \equiv \sum_1^q B_j - \sum_1^p A_j \geqslant 0.$$

We now replace α by αt_2 in (3.1), multiply both sides by $t_2^{\rho_2 - 1} \exp(-t_2)$ and integrate with respect to t_2 from 0 to ∞ to get I_2. Proceeding in this way we get

$$I_r = \int_0^{\infty} \cdots \int_0^{\infty} \prod_{j=1}^{r} \left\{ t_j^{\rho_j - 1} \exp(-t_j) \right\}$$

$$\times H_{p,q}^{m,n}\left[\alpha t_r \cdots t_1 \,\middle|\, \begin{matrix} \{(a_p, A_p)\} \\ \{(b_q, B_q)\} \end{matrix} \right] dt_1 \cdots dt_r$$

$$= H_{p+r,q+r}^{m,n+r}\left[\alpha \,\middle|\, \begin{matrix} \{(1-\rho_r, 1)\}, \{(a_p, A_p)\} \\ \{(b_q, B_q)\} \end{matrix} \right] \qquad \ldots(3.2)$$

provided $\rho_j + \min \operatorname{Re}(b_h/B_h) > 0$, $(h = 1,\ldots, m)$, $(j = 1,\ldots, r)$ and $\delta \geqslant 0$.

Hence from (3.2) we formulate the pdf as $f(t_1,\ldots,t_r)$

$$= \frac{\prod_{j=1}^{r} \left\{ t_j^{\rho_j - 1} \exp(-t_j) \right\} H_{p,q}^{m,n}\left[\alpha\, t_r, \ldots, t_1 \,\middle|\, \begin{matrix} \{(a_p, A_p)\} \\ \{(b_q, B_q)\} \end{matrix} \right]}{H_{p+r,q}^{m,n+r}\left[\alpha \,\middle|\, \begin{matrix} \{(1-\rho_r, 1)\}, \{(a_p, A_p)\} \\ \{(b_q, B_q)\} \end{matrix} \right]}$$

for $t_j > 0$, $(j = 1,\ldots, r)$

$= 0$, otherwise. ...(3.3)

The characteristic function for the density (3.3), as defined by (2.2) can be immediately evaluated and seen to be $\phi(w_1,\ldots, w_r)$

$$= \frac{H_{p+r,q}^{m,n+r}\left[\dfrac{\alpha}{\prod_{j=1}^{r}(1 - iw_j)} \,\middle|\, \begin{matrix} \{(1-\rho_r, 1)\}, \{(a_p, A_p)\} \\ \{(b_q, B_q)\} \end{matrix} \right]}{\prod_{j=1}^{r}(1 - iw_j)^{\rho_j} \; H_{p+r,q}^{m,n+r}\left[\alpha \,\middle|\, \begin{matrix} \{(1-\rho_r, 1)\}, \{(a_p, A_p)\} \\ \{(b_q, B_q)\} \end{matrix} \right]}. \quad \ldots(3.4)$$

It is to be emphasised here that the parameters involved in (3.3) are to be so chosen that the pdf $f(t_1,\ldots, t_r)$ is everywhere non-negative. It can be seen that this pdf satisfies the condition (2.1).

§4. We conclude this paper by showing one reduction of the pdf introduced in section 3.

In the pdf given by (3.3) let $r = 2, m = 1, n = 0, p = 0, q = 1, b_1 = b$ and $B_1 = 1$ then we have the bivariate pdf :

$$f(t_1, t_2) = \frac{\alpha^b\, t_1^{\rho_1 + b - 1}\, t_2^{\rho_2 + b - 1}\, e^{-t_1 - t_2 - \alpha t_1 t_2}}{G_{2,1}^{1,2}\left(\alpha \,\middle|\, \begin{matrix} 1-\rho_1, 1-\rho_2 \\ b \end{matrix} \right)} \quad \text{for } t_1, t_2 > 0$$

$= 0$, otherwise

and the characteristic function, from (3.4), will be

$$\phi(w_1, w_2) = \frac{G_{2,1}^{1,2}\left(\dfrac{\alpha}{(1 - iw_1)(1 - iw_2)} \,\middle|\, \begin{matrix} 1-\rho_1, 1-\rho_2 \\ b \end{matrix} \right)}{(1 - iw_1)^{\rho_1} (1 - iw_2)^{\rho_2} \; G_{2,1}^{1,2}\left(\alpha \,\middle|\, \begin{matrix} 1-\rho_1, 1-\rho_2 \\ b \end{matrix} \right)}$$

for the set of r. v. (X_1, X_2).

When necessary, the G-functions here may be expressed in terms of Whittaker function $w_{\lambda,\mu}(.)$

Further, with $r = 1$, $m = 1$, $n = 0$, $p = 0$, $q = 1$, $b_1 = b$, $B_1 = 1$ we get from (3.3) and (3.4), after a simplification of the G-function involved,

$$f(t) = \frac{(1+\alpha)^{b+\rho}\, t^{b+\rho-1}\, e^{-(1+\alpha)t}}{\Gamma(b+\rho)}, \quad t > 0$$

$$= 0, \text{ otherwise}$$

with the c. f. as $\phi(w) = \dfrac{(1+\alpha)^{b+\rho}}{(1+\alpha - iw)^{b+\rho}}$

as a single-variate pdf together with its characteristic function.

This futher reduces, when we take $b + \rho = 1$ and $1 + \alpha = \lambda$, to the well-known exponential probabilty law

$$f(t)\, \lambda e^{-\lambda t}, \quad t > 0$$

$$= 0, \text{ otherwise}$$

with the characteristic function

$$\phi(w) = \frac{\lambda}{\lambda - iw}.$$

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