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THE FOURIER TRANSFORM APPROACH TO GENERAL *B*-SPLINES

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In this paper, a theory of *B*-splines from the Fourier transform approach is developed extending the results of Schoenberg to general splines, including the case of multiple knots. Some new results are established and most of the well known results like the recurrence formulae for splines, are derived using the Fourier transform representation of the general splines.

§1. Since Schoenberg (1946) introduced spline functions, many different representations of the basic splines, called the *B*-splines and their properties have been given. The equidistance case, defining uniform splines, has been extended to non-uniform splines where the data is collected at arbitrary knots. But in all these extensions and generalizations, the original definition of the basic splines, as a map under the complex Fourier transform, has been ignored, in favour of definition involving mostly divided differences (de Boor 1976, Curry and Schoenberg 1966, Titchmarsh 1948).

In this paper, we develop a theory of *B*-splines from the Fourier transform approach. First we define a function $g_{k,l}(x)$ on monotonically increasing sequence of points $\{t_\nu\}$, called knots, satisfying a recursive relationship involving divided differences. This function is shown to be a complex valued function of a real variable x in $-\infty < x < \infty$, and among other properties, it is proved that its modulus lies inside the unit circle. Then the complex Fourier transform of the function $g_{k,l}(x)$, is a real valued function, which defines general *B*-spline function $C_{k,l}(x)$ at arbitrary knots. This representation of spline does not involve divided differences and is a natural extension of the original definition of the uniform spline introduced by Schoenberg. By using this definition of general splines, we have established a number of well-known results and some new results. Also to illustrate the general nature of our results, we have derived some of the more common results, as special cases.

§2. In this section we shall define the function $g_{k,l}(u)$, a complex valued function of the real variable u, where $-\infty < u < \infty$, and establish some of its most important properties.

Definition 2.1—Given a non-decreasing sequence of real numbers $\{t_\nu\}$, $t_\nu \leqq t_{\nu+1}$, $-\infty < \nu < \infty$, we define

$$g_{k,l}(u) \equiv g_{k,l}(u; t_l, t_{l+1}, \ldots, t_{l+k}), \quad -\infty < u < \infty$$

satisfying the recurrence formula,

$$g_{k,l}(u) = \frac{k}{(t_{l+k} - t_l)\, iu} \{g_{k-1,\, l+1}(u) - g_{k-1,l}(u)\} \qquad \ldots(2.1)$$

$$g_{0,l}(u) = g_{0,l}(u) = e^{iut_l} \text{ if } t_l = t_{l+k}. \qquad \ldots(2.2)$$

Definition 2.2—Let $t_l \leqq t_{l+1} \leqq \ldots \leqq t_{l+k}$; then,

$$\bar{\tau} \equiv \bar{\tau}_{\nu,k,l} = \frac{\nu!\, k!}{(\nu + k)!}\, \tau_{\nu,k,l} \qquad \ldots(2.3)$$

where

$$\tau_{\nu,k,l} = \begin{cases} 0 & \text{if } \nu = -k, -k+1, \ldots, -1 \\ 1 & \text{if } \nu = 0 \\ \displaystyle\sum_{0 \leqq j_1 \leqq \cdots \leqq j_\nu \leqq k} \prod_{\mu=1}^{\nu} t_{l+j_\mu} \end{cases}$$

Lemma 2.1— $\bar{\tau}_{\nu,k,l} = \dfrac{k}{(\nu+1)(t_{l+k} - t_l)} (\bar{\tau}_{\nu+1,k-1,l+1} - \bar{\tau}_{\nu+1,k-1,l})$ $\qquad \ldots(2.4)$

where $\bar{\tau}$ is given as in Definition 2.2 above.

PROOF: Let $\nu = 0$. Then by Definition 2.2 one can easily show that

$$\bar{\tau}_{1,k-1,\, l-1} - \bar{\tau}_{1,k-1,l} = \frac{1}{k}(t_{l+k} - t_l).$$

Hence

$$\frac{k}{t_{l+k} - t_l} (\bar{\tau}_{1,k-1,l-1} - \bar{\tau}_{1,\, k-1,l}) = 1 = \bar{\tau}_{0,k,l}$$

as required. For $\nu = 1, 2, \ldots$, consider the product

$$\tau_{\nu+1,k-1,l+1} = \sum_{0 \leqq j_1 \leqq \cdots \leqq j_{\nu+1} \leqq k-1} \prod_{\mu=1}^{\nu+1} t_{l+1+j_\mu} = \sum_{1 \leqq j_1 \leqq \cdots \leqq j_{\nu+1} \leqq k} \times \prod_{\mu=1}^{\nu+1} t_{l+j_\mu}$$

$$= \Big(\sum_{j_{\nu+1}=k,\, 1 \leqq j_1 \leqq \cdots \leqq j_\nu \leqq k} + \sum_{1 \leqq j_1 \leqq \cdots \leqq j_{\nu+1} \leqq k-1} \Big) \prod_{\mu=1}^{\nu+1} t_{l+j_\mu}$$

$$= \Big(\sum_{j_{\nu+1}=k,\, 0 \leqq j_1 \leqq \cdots \leqq j_\nu \leqq k} - \sum_{j_{\nu+1}=k,\, j_1=0,\, 0 \leqq j_2 \leqq \cdots \leqq j_\nu \leqq k} + \sum_{1 \leqq j_1 \leqq \cdots \leqq j_{\nu+1} \leqq k-1} \Big) \Big(\prod_{\mu=1}^{\nu+1} t_{l+j_\mu} \Big). \qquad \ldots(2.5)$$

Similarly,

$$\tau_{\nu+1,k-1,l} = \sum_{0\leqq j_1\leqq\cdots\leqq j_{\nu+1}\leqq k-1} \prod_{\mu=1}^{\nu+1} t_{l+j_\mu}$$

$$= \Big(\sum_{\substack{j_1=0 \\ 0\leqq j_2\leqq\cdots\leqq j_{\nu+1}\leqq k}} - \sum_{\substack{j_1=0, j_{\nu+1}=k \\ 0\leqq j_2\leqq\cdots\leqq j_\nu\leqq k}} + \sum_{1\leqq j_1\leqq\cdots\leqq j_{\nu+1}\leqq k-1} \Big)$$

$$\times \prod_{\mu=1}^{\nu+1} t_{l+j_\mu} . \qquad \ldots(2.6)$$

Now from (2.5) and (2.6), we have

$$\tau_{\nu+1,k-1,l+1} - \tau_{\nu+1,k-1,l} = \Big(\sum_{\substack{j_{\nu+1}=k \\ 0\leqq j_1\leqq\cdots\leqq j_\nu\leqq k}} - \sum_{\substack{j_1=0 \\ 0\leqq j_2\leqq\cdots\leqq j_{\nu+1}\leqq k}} \Big) \prod_{\mu=1}^{\nu+1} t_{l+j_\mu}$$

$$= t_{l+k} \sum_{0\leqq j_1\leqq\cdots\leqq j_\nu\leqq k} \prod_{\mu=1}^{\nu} t_{l+j_\mu}$$

$$- t_l \sum_{0\leqq j_1\leqq\cdots\leqq j_\nu\leqq k} \prod_{\mu=1}^{\nu} t_{l+j_\mu}$$

$$= (t_{l+k} - t_l)\, \tau_{\nu,k,l} . \qquad \ldots(2.7)$$

Hence

$$\frac{k}{(\nu+1)(t_{l+k}-t_l)} (\bar{\tau}_{\nu+1,k-1,l+1} - \bar{\tau}_{\nu+1,k-1,l})$$

$$= \frac{k}{(\nu+1)(t_{l+k}-t_l)} \frac{(\nu+1)!\,(k-1)!}{(\nu+k)!} (\tau_{\nu+1,k-1,l+1} - \tau_{\nu+1,k-1,l})$$

$$= \frac{\nu!\,k!}{(\nu+k)!}\ \tau_{\nu,k,l} = \bar{\tau}_{\nu,k,l}$$

using (2.7) and (2.3).

Lemma 2.2—Let $t_l \leqq t_{l+1} \leqq \ldots \leqq t_{l+k}$. Then

$$\bar{\tau}_{\nu,k,l} = \frac{k}{(\nu+k)(t_{l+k}-t_l)} \{t_{l+k}\, \bar{\tau}_{\nu,k-1,l+1} - t_l\, \bar{\tau}_{\nu,k-1,l}\} \qquad \ldots(2.8)$$

where $\bar{\tau}$ is defined by (2.3).

PROOF : By Definition 2.2

$$\tau_{\nu,k,l} = \sum_{0\leqq j_1\leqq\cdots\leqq j_\nu\leqq k} \prod_{\mu=1}^{\nu} t_{l+j_\mu} = t_l \sum_{0\leqq j_1\leqq\cdots\leqq j_{\nu-1}\leqq k} \prod_{\mu=1}^{\nu-1} t_{l+j_\mu}$$

$$+ \sum_{1\leqq j_1\leqq\cdots\leqq j_\nu\leqq k} \prod_{\mu=1}^{\nu} t_{l+j_\mu} \qquad \ldots(2.9)$$

Also write

$$\tau_{\nu,k,l} = t_{l+k} \sum_{0\leqq j_1\leqq\cdots\leqq j_{\nu-1}\leqq k} \prod_{\mu=1}^{\nu-1} t_{l+j_\mu} + \sum_{0\leqq j_1\leqq\cdots\leqq j_\nu\leqq k-1} \prod_{\mu=1}^{\nu} t_{l+j_\mu}. \qquad \ldots(2.10)$$

Now using the representation (2.9) and (2.10) for $\tau_{\nu,k,l}$ respectively,

$$(t_{l+k} - t_l)\,\tau_{\nu,k,l} = t_{l+k} \sum_{1\leqq j_1\leqq\cdots\leqq j_\nu\leqq k} \prod_{\mu=1}^{\nu} t_{l+j_\mu} - t_l \sum_{0\leqq j_1\leqq\cdots\leqq j_\nu\leqq k-1} \times \prod_{\mu=1}^{\nu} t_{l+j_\mu} = t_{l+k}\,\tau_{\nu,k-1,l+2} - t_l\,\tau_{\nu,k-1,\,l}.$$

Hence

$$\begin{aligned}\bar{\tau}_{\nu,k,l} &= \frac{\nu!\,k!}{(\nu+k)!}\,\tau_{\nu,k,l} \\ &= \frac{\nu!\,k!}{(\nu+k)!\,(t_{l+k}-t_l)}\,(t_{l+k}\,\tau_{\nu,k-1,l+1} - t_l\,\tau_{\nu,k-1,l}). \\ &= \frac{k}{(\nu+k)\,(t_{l+k}-t_l)}\,(t_{l+k}\,\bar{\tau}_{\nu,k-1,l+1} - t_l\,\bar{\tau}_{\nu,k-1,l})\end{aligned}$$

as required.

Theorem 2.1—If $t_l \leqq t_{l+1} \leqq \ldots \leqq t_{l+k}$, and $\bar{\tau}_\nu$ as defined above, then

$$g_{k,l}(u) = 1 + \sum_{\nu=1}^{\infty} \frac{(iu)^\nu}{\nu!}\,\bar{\tau}_{\nu,k,l}. \qquad \ldots(2.11)$$

PROOF (by induction): If $k = 0$, then using (2.2),

$$g_{0,l}(u) = e^{iut_l} = \sum_{\nu=0}^{\infty} \frac{(iu)^\nu}{\nu!}\,(t_l)^\nu = 1 + \sum_{\nu=1}^{\infty} \frac{(iu)^\nu}{\nu!}\,\bar{\tau}_{\nu,0,l}$$

since

$$\bar{\tau}_{\nu,0,l} = \tau_{\nu,0,l} = \sum_{0\leqq j_1\leqq\cdots\leqq j_\nu\leqq 0} \prod_{\mu=1}^{\nu} t_{l+j_\mu} = t_l^\nu.$$

Suppose (2.11) holds for $k-1$. Then using the recurrence formula (2.1),

$$\begin{aligned}g_{k,l}(u) &= \frac{k}{(t_{l+k}-t_l)\,iu}\,\{g_{k-1,l+1}(u) - g_{k-1,l}(u)\} \\ &= \frac{k}{(t_{l+k}-t_l)\,iu} \sum_{\nu=1}^{\infty} \frac{(iu)^\nu}{\nu!}\,(\bar{\tau}_{\nu,k-1,l+1} - \bar{\tau}_{\nu,k-1,l})\end{aligned}$$

(*equation continued on p.* 815)

$$= \sum_{\nu=0}^{\infty} \frac{(iu)^{\nu}}{(\nu+1)!} \frac{k}{t_{l+k} - t_l} (\bar{\tau}_{\nu+1,k-1,l+1} - \bar{\tau}_{\nu+1,k-1,l})$$

$$= \sum_{\nu=0}^{\infty} \frac{(iu)^{\nu}}{\nu!} \bar{\tau}_{\nu,kl}$$

using the result (2.4) of Lemma 2.1. Hence

$$g_{k,l}(u) = 1 + \sum_{\nu=1}^{\infty} \frac{(iu)^{\nu}}{\nu!} \bar{\tau}_{\nu,k-1}.$$

as required

Next we shall prove some results giving the bounds of the function $g_{k,l}(u)$.

Lemma 2.3—If $t_l \leqq t_{l+1} \leqq \ldots \leqq t_{l+k}$, then $g_{k,l}(u)$ is uniformly bounded, that is

$$| g_{k,l}(u) | < A \qquad \ldots(2.12)$$

for some constant A and for all u.

PROOF : Now

$$g_{k,l}(u) = 1 + \sum_{\nu=1}^{\infty} \frac{(iu)^{\nu}}{\nu!} \bar{\tau}_{\nu} = 1 + \sum_{\nu=1}^{\infty} \frac{\left(iu\tau_{\nu}^{*}\right)^{\nu}}{\nu!}$$

where $\left(\tau_{\nu}^{*}\right)^{\nu} = \bar{\tau}_{\nu}$, and $t_l \leqq \tau_{\nu}^{*} \leqq t_{l+k}$, and $\lim_{n \to \infty} \tau_n^{*} = t_{l-k}$.

Thus,

$$g_{k,l}(u) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\tau_{2n}^{*} u\right)^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n \left(\tau_{2n+1}^{*} u\right)^{2n+1}}{(2n+1)!}.$$

It is an easy matter to see that both the series are uniformly convergent for all u; therefore

$$\left| g_{k,l}(u) \right|^2 = \left(\sum_{n=0}^{\infty} \frac{(-1)^n \left(\tau_{2n}^{*} u\right)^{2n}}{(2n)!} \right)^2$$

$$+ \left(\sum_{n=0}^{\infty} \frac{(-1)^n \left(\tau_{2n+1}^{*} u\right)^{2n+1}}{(2n+1)!} \right)^2 \leqq A.$$

for some constant A, proving the lemma.

We shall later improve this estimate (Theorem 3.5) and in fact show that $| g_{k,l}(u) | \leqq 1$. This is an important property of the function $g_{k,l}(u)$ in the sense that the absolute value of this complex valued function lies inside the unit circle.

Lemma 2.4—If $\bar{\tau}_{\nu,k,l}$ is given by (2.3), then

$$\sigma^2 \equiv \bar{\tau}_{2,k,l} - \bar{\tau}^2_{1,k,l} = \frac{1}{(k+1)^2(k+2)} \sum_{0 \leqq j_1 \leqq j_2 \leqq k} (t_{l+j_1} - t_{l+j_2})^2 \qquad \ldots(2.13)$$

PROOF : By definition 2.2

$$\sigma^2 = \frac{2}{(k+2)(k+1)} \sum_{0 \leqq j_1 \leqq j_2 \leqq k} t_{l+j_1} t_{l+j_2} - \left(\frac{1}{k+1} \sum_{0 \leqq j_1 \leqq k} t_{l+j_1} \right)^2 = \frac{1}{(k+2)(k+1)^2} \left\{ k \sum_{0 \leqq j_1 \leqq j_2 \leqq k} - (k+2) \sum_{0 \leqq j_2 \leqq j_1 \leqq k} \right\} t_{l+j_1} t_{l+j_2} = \frac{1}{(k+2)(k+1)^2} \times \left\{ k \sum_{0 \leqq j_1 \leqq j_2 \leqq k} - 2 \sum_{0 \leqq j_1 \leqq j_2 \leqq k} \right\} t_{l+j_1} t_{l+j_2}$$

$$= \frac{1}{(k+2)(k+1)} \sum_{0 \leqq j_1 \leqq j_2 \leqq k} \left(t_{l+j_1} - t_{l+j_2} \right)^2$$

as required.

Theorem 2.2— $| g_{k,l}(u) |^2 = 1 - \sigma^2 u^2 + O(u^4)$, as $u \to 0$ $\qquad \ldots(2.14)$

and as $u \to \infty$,

$$g_{k,l}(u) = \begin{cases} O(u^{-1}) & \text{if } t_l = \ldots = t_{l+k-1} < t_{l+k} \\ & \text{or } t_l < t_{l+1} = \ldots = t_{l+k} \\ O(u^{r-k-1}) & \text{if } t_l = \ldots = t_{l+j-1} < t_{l+j} = \ldots = t_{l+k} \\ & \text{or } j \neq 1 \text{ or } k \\ O(1) & \text{if } r = k. \end{cases} \qquad \ldots(2.15)$$

where σ^2 is defined by (2.13) and r is the number of equalities in the sequence $t_l \leqq t_{l+1} \leqq \ldots \leqq t_{l+k}$.

PROOF : By (2.12),

$$g_{k,l}(u) = 1 + \sum_{\nu=1}^{\infty} \frac{(iu)^\nu}{\nu!} \bar{\tau}_{\nu,k,l} = 1 - \frac{1}{2} u^2 \bar{\tau}_{2,k,l} + O(u^4) + iu(\bar{\tau}_{1,k,l} + O(u^4))$$

as $u \to 0$. Hence

$$| g_{k,l}(u) |^2 = 1 - u^2 (\bar{\tau}_2 - \bar{\tau}_1^2) + O(u^4)$$
$$= 1 - \sigma^2 u^2 + O(u^4) \quad \text{as } u \to 0.$$

Now to prove (2.15), we shall consider two cases; $r = k$ and $r < k$. If $r = k$, then $t_l = t_{l+k}$ and from (2.3), we have

$$\left| g_{k,j}(u) \right| = \left| e^{iut_l} \right| = 1.$$

If $r < k$, let us first consider the case when

$$t_l = \ldots = t_{l+k-1} < t_{l+k}.$$

Let $k = 1$, then by the recurrence formula, we have

$$g_{1,l}(u) = \frac{1}{(t_{l+1} - t_l)\, iu} \{g_{0,l+1}(u) - g_{0,l}(u)\}$$
$$= \frac{1}{(t_{l+1} - t_l)\, iu} \{e^{iut_{l+1}} - e^{iut_l}\}.$$

Hence

$$| g_{1,l}(u) | = O(u^{-1}), \text{ as } u \to \infty.$$

Suppose

$$g_{k-1,l+1}(u) = O(u^{-1}).$$

Then by induction, using the recurrence formula

$$g_{k,l}(u) = \frac{k}{(t_{l+k} - t_l)\, iu} \{g_{k-1,l+1}(u) - g_{k-1,l}(u)\}.$$

$$= \frac{k}{(t_{l+k} - t_l)\, iu} \{g_{k-1,l+1}(u) - e^{iut_l}\}$$

And

$$g_{k,l}(u) = O(u^{-1}) \{O(u^{-1}) + O(1)\} = O(u^{-1}), \text{ as } u \to \infty.$$

If $t_l < t_{l+1} = \ldots = t_{l+k}$, then

$$g_{k,l}(u) = \frac{k}{(t_{l+k} - t_l)\, iu} \{e^{iut_{l+1}} - g_{k-1,l}(u)\}.$$

Suppose $g_{k-1,l}(u) = O(u^{-1})$, as $u \to \infty$, then

$$g_{k,l}(u) = O(u^{-1}), \text{ as } u \to \infty.$$

Next consider the case when $r < k$ and

$$t_l = \ldots = t_{l+j-1} < t_{l+j} = \ldots = t_{l+k}, \; j \neq 1 \text{ or } k.$$

Let us assume that

$$g_{k-1,l+1}(u) = O\left(|u|^{r_{k-1,l+1}-k}\right), \quad u \to \infty$$

and

$$g_{k-1,l}(u) = O\left(|u|^{r_{k-1,l}-k}\right), \quad u \to \infty.$$

where $r_{k-1,l+1}$ and $r_{k-1,l}$ denote the number of equalities in the sequences $t_{l+1} \leqq t_{l+2} \leqq \ldots \leqq t_{l+k}$ and $t_l \leqq \ldots \leqq t_{l+k-1}$, respectively. Then from

$$g_{k,l}(u) = \frac{k}{(t_{l+k} - t_l)\, iu} \{g_{k-1,l+1}(u) - g_{k-1,l}(u)\}.$$

we have,

$$|g_{k,l}(u)| = O(u^{-1})\{O(u^{r_{k-1,l+1}-k}) + O(u^{r_{k-1,l}-k})\} = O(u^{r-k-1}),$$

as $u \to \infty$

where $r \geqq$ more $(r_{k-1,l+1}, r_{k-1,l})$, hence the theorem.

Corollary—If $t_l < t_{l+k}$, then $g_{k,l}(u) \to O$ as $u \to \infty$. And if

$$g_{k,l}(u) = e^{iut_l},$$

then

$$g_{\nu,l}(u) = e^{iut_l}, \quad \text{for } \nu \leqq k.$$

§3. In this section we shall define the Fourier transform and establish some important properties of it.

Theorem 3.1— Let $t_l < t_{l+k}$. Then the Fourier transform of the function $g_{k,l}(u)$ exists and is defined by

$$C_{k,l}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g_{k,l}(u)\, e^{-inx}\, du. \qquad \ldots(3.1)$$

Further

$$C_{k,l}(x) \in C^{k-2}(t_l, \infty)$$

if $t_l = \ldots = t_{l+k-1} < t_{l+k}$ or $t_l < t_{l+1} = \ldots = t_{l+k}$,

and

$$C_{k,l}(x) \in C^{k-1-r}$$

if $t_l = \ldots = t_{l+j-1} < t_{l+j} = \ldots = t_{l+k}$, $j \neq 1$ or k.

Here, as before, r denotes the number of equalities in the non-decreasing sequence $\{t_{\iota+j}\}$, $j = 1, 2, \ldots, k$.

PROOF : If $t_\iota < t_{\iota+k}$, then clearly $r < k$, or $r \leqq k-1$. Let $r \leqq k-2$. Then from (2.15) it is obvious that

$$| g_{k,\iota}(u) | = O(u^{-2}), \text{ as } u \to \infty.$$

Hence, $g_{k,\iota}(u) \in L(-\infty, \infty)$ and the integral in (3.1) exists and in fact is absolutely convergent, thus defining the function $C_{k,\iota}(x)$.

Now let $r = k - 1$. If

$$t_\iota = \ldots = t_{\iota+k-1} < t_{\iota+k},$$

then from the definition

$$g_{0,\iota}(u) = g_{1,\iota}(u) = \ldots = g_{k-1,\iota}(u) = e^{iut_\iota}$$

and

$$\begin{aligned} g_{k,\iota}(u) &= \frac{k}{(t_{\iota+k} - t_\iota)\, iu} \{g_{k-1,\iota+1}(u) - g_{k-1,\iota}(u)\} \\ &= \frac{k}{(t_{\iota+k} - t_\iota)\, iu} \{g_{k-1,\iota+1}(u) - e^{iut_\iota}\} \\ &= \frac{-ke^{iut_\iota}}{t_{\iota+k} - t_\iota)\, iu} - \ldots - \frac{k(k-1)\, e^{iut_\iota}}{(t_{\iota+k} - t_\iota)^2 (iu)^2} \\ &\quad - \ldots - \frac{k(k-1) \ldots 2}{(t_{\iota+k} - t_\iota)^k (iu)^k} \; g_{1,\iota+k-1}(u). \\ &= -e^{iut_\iota} \left\{ \frac{k}{(t_{\iota+k} - t_\iota)\, iu} + \frac{k(k-1)}{(t_{\iota+k} - t_\iota)^2 (iu)^2} \right. \\ &\quad \left. + \ldots + \frac{k!}{(t_{\iota+k} - t_\iota)^k (iu)^k} \right\} + \frac{k!\, e^{iut_{\iota+k}}}{(t_{\iota+k} - t_\iota)^k (iu)^k}. \end{aligned}$$

Notice that the right-hand side of the above equation can be obtained as a result of integration by parts of the integral

$$\frac{k}{(t_{\iota+k} - t_\iota)^k} \int_{t_\iota}^{t_{\iota+k}} (t_{\iota+k} - x)^{k-1} e^{iux}\, dx = \int_{t_\iota}^{t_{\iota+k}} f_{k,\iota}(x) e^{iux}\, dx \text{ say}$$

where clearly $f_{k,\iota}(x) \in C^{k-2}(t_\iota, \infty)$. Whence, $f_{k,\iota}(x)$ has the inverse Fourier transform (Titchmarsh 1948), and is given by

$$f_{k,\iota}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g_{k,\iota}(u_\iota)\, e^{iux}\, du,$$

the integral exists in the Cauchy's sense.

Similarly if

$$t_i < t_{i+1} = \ldots = t_{i+k}.$$

we can show that Fourier transform of $g_{k,i}(x)$ exists and is given by

$$\tilde{f}_{k,i}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g_{k,i}(u)\, e^{-iux} dx$$

and that $\tilde{f}_{k,i}(x) \in C^{k-2}(t_i, \infty)$.

Hence if $r = k - 1$, then there exists a function $C_{k,i}(x)$ defined by

$$C_{k,i}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g_{k,i}(u)\, e^{-iux}\, du$$

belonging to $C^{k-2}(t_i, \infty)$.

Next, we shall prove that $C_{k,i}(x) \in C^{k-1-r}$, when

$$t_i = \ldots = t_{i+j-1} < t_{i+j} = \ldots t_{i+k},\ j \neq 1 \text{ or } k.$$

From (2.15), we have

$$| g_{k,i}(u) | = O(u^{r-k-1}) \text{ as } u \to \infty.$$

Therefore

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} (iu)^{k-r-1}\, g_{k,i}(u)\, e^{-iux}\, du = C_{k,i}^{(k-r-1)}(x).$$

The integral being absolutely and uniformly convergent for all x. Hence

$$C_{k,i}(x) \in C^{k-1-r}(-\infty, \infty)$$

as required, which proves the theorem.

Note that if $k = 0$, we shall define

$$C_{0,k}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g_{0,i}(u)\, e^{-iux}\, du$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iu(x-t_i)}\, du \equiv \delta(x - t_i).$$

where δ is the Dirac delta function.

Theorem 3.2—Let $\nu = 0, 1, 2, \ldots$ and $j = 0, 1, 2, \ldots$ Then

$$\int_{-\infty}^{\infty} x^{\nu} C_{k,l}^{(j)}(x)\, dx = \frac{(-1)^j \nu!}{(\nu - j)!} \bar{\tau}_{\nu-j} \qquad \ldots(3.2)$$

where $C_{k,l}$ and $\bar{\tau}$ are defined above.

PROOF : From Theorem 3.1, we have

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} g_{k,l}(u)\, e^{-iux}\, dx = C_{k,l}(x) \in C^{k-1-r}(-\infty, \infty).$$

Therefore for every $j \leqq k - 1 - r$,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} (-iu)^j g_{k,l}(j)\, e^{-iux}\, dx = C_{k,l}^{(j)}(x),$$

whence by Fourier inversion,

$$(-iu)^j g_{k,l}(u) = \int_{-\infty}^{\infty} C_{k,l}^{(j)}(x)\, e^{iux}\, dx.$$

or, using (2.11),

$$(-iu)^j \sum_{\nu=0}^{\infty} \frac{(iu)^{\nu}}{\nu!} \bar{\tau} = \int_{-\infty}^{\infty} C_{k,l}^{(j)}(x) \sum_{\nu=0}^{\infty} \frac{(iux)^{\nu}}{\nu!}\, dx$$

and

$$(-iu)^j \sum_{\nu=j}^{\infty} \frac{(iu)^{\nu}}{(\nu - j)!} \bar{\tau}_{\nu-j} = \sum_{\nu=0}^{\infty} \frac{(iu)^{\nu}}{\nu!} \int_{-\infty}^{\infty} C_{k,l}^{(j)}(x)\, x^{\nu} dx.$$

Now since $\bar{\tau}_{-1} = \bar{\tau}_{-2} = \ldots = \bar{\tau}_{-k} = 0$, we have

$$(-1)^j \sum_{\nu=0}^{\infty} \frac{(iu)^{\nu}}{(\nu - j)!} \bar{\tau}_{\nu-j} = \sum_{\nu=0}^{\infty} \frac{(iu)^{\nu}}{\nu!} \int_{-\infty}^{\infty} C_{k,l}^{(j)}(x)\, x^{\nu} dx.$$

Hence,

$$\frac{(-1)^j}{(\nu - j)!} \bar{\tau}_{\nu-j} = \frac{1}{\nu!} \int_{-\infty}^{\infty} C_{k,l}^{(j)} x^{\nu} dx$$

as required.

The result (3.2) gives us a formula for the general moments of the function $C_{k,l}(x)$ and its derivatives

Corollaries—(i) $$\int_{-\infty}^{\infty} C_{k,l}(x)\,dx = 1. \qquad \ldots(3.3)$$

(ii) $$\int_{-\infty}^{\infty} C_{k,l}^{(j)}(x)\,dx = 0, \text{ when } 0 < j \leqq k-1-r.$$

(iii) $$\int_{-\infty}^{\infty} x^{\nu} C_{k,l}(x)dx = \bar{\tau}_{\nu}. \qquad \ldots(3.4)$$

Next we state a well known result for reference.

Lemma 3.1—If $\left\langle x^{\nu}, f(x) \right\rangle = \int_a^b x^{\nu} f(x)\,dx = 0,$

for all ν and $f \in C(a, b)$, then $f(x) \equiv 0$ for *a. a* x in (a, b). Here we define the inner product as

$$\left\langle x^{\nu}, f(x) \right\rangle = \int_{-\infty}^{\infty} x^{\nu} f(x)dx \ldots$$

Lemma 3.2—A function $f(x) \in C(-\infty, \infty)$ has compact support if there exists a function $g(x) \in L^2(a, b)$, and $-\infty < a \leqq b < \infty$, such that for each $\nu = 0, 1, 2, \ldots$

$$\left\langle x^{\nu}, f(x) \right\rangle = \int_a^b x^{\nu} g(x)\,dx$$

PROOF : Suppose $f(x)$ has a compact support, that is

$$f(x) = 0 \text{ if } x \notin (a, b).$$

Then

$$\left\langle x^{\nu}, f(x) \right\rangle = \int_a^b x^{\nu} f(x)\,dx$$

$$= \int_a^b x^{\nu} g(x)\,dx$$

by assumption. Hence

$$f(x) = g(x) \in L^2(0, \infty).$$

Now let $g(x) \in L^2(0, \infty)$.

Define the function

$$F(x) = \begin{cases} g(x) \text{ if } x \in (a, b) \\ 0, \text{ otherwise} \end{cases}$$

then

$$\left\langle x^{\nu}, f(x) \right\rangle = \int_{a}^{b} x^{\nu} g(x)\, dx$$

$$= \int_{-\infty}^{\infty} x^{\nu} F(x)\, dx$$

or,

$$\int_{-\infty}^{\infty} x^{\nu} [f(x) - F(x)]\, dx = 0, \quad \nu = 0, 1, 2, \ldots$$

and

$$\int_{-\infty}^{\infty} e^{-x^2} x^{\nu} [f(x) - F(x)]\, e^{x^2} dx = 0$$

therefore

$$< x^{\nu}, e^{x^2} (f - F) > = 0$$

in the sense of $L^2\{(-\infty, \infty); e^{-x^2}\}$. But $\{x^{\nu}\}$ is a basis in $L^2\{(-\infty, \infty); e^{-x^2}\}$; hence there are only two points of discontinuity, a and b. If $g(x) \in C(a, b)$, then due to Lemma 3.1,

$$f(x) = F(x) \qquad a.e$$

proving this lemma.

Theorem 3.3—If $t_l < t_{l+k}$, then there exists a function $q_{k,l}(x)$, such that

$$\bar{\tau}_{\nu,k,l} = \int_{t_l}^{t_{l+k}} x^{\nu} q_{k,l}(x)\, dx, \quad \nu = 0,1,2, \qquad \ldots(3.5)$$

PROOF (by induction) : Let $k = 1$. Then by (2.3)

$$\tau_{\nu,1,l} = \begin{cases} 0 \text{ if } \nu = 0 \\ \displaystyle\sum_{0 \leqq j_1 \leqq \cdots \leqq j_{\nu} \leqq 1} \prod_{\mu=1}^{\nu} t_{l+j_{\mu}} \end{cases}$$

$$= t_{l+1}^{\nu} + t_{l+1}^{\nu+1} t_l + \ldots + t_{l+1} t_l^{\nu-1} + t_l^{\nu}$$

$$= \frac{1}{t_{l+1} - t_l} \left(t_{l+1}^{\nu+1} - t_l^{\nu+1} \right)$$

$$= \frac{\nu + 1}{t_{l+1} - t_l} \int_{t_l}^{t_{l+1}} x^{\nu}\, dx, \quad \nu = 1, 2, 3, \ldots \text{ and } 0.$$

Hence

$$\bar{\tau}_{\nu,1l} = \int_{tl}^{t_{l+1}} x^{\nu}\, q_{1,l}(x)\, dx$$

where

$$q_{1,l}(x) = \frac{\nu+1}{t_{l+1}-t_l} \text{ and } \nu = 0, 1, 2, \ldots$$

Now suppose that the formula (3.5) holds for $k-1$ and $\nu = 0, 1, 2, \ldots$; then using the recurrence formula (2.4)

$$\begin{aligned}\bar{\tau}_{\nu,k,l} &= \frac{k}{(\nu+1)(t_{l+k}-t_l)}\left(\bar{\tau}_{\nu+1,k-1,l+1} - \bar{\tau}_{\nu+1,k-1,l}\right) \\ &= \frac{k}{(\nu-1)(t_{l+k}-t_l}\left(\int_{t_{l+1}}^{t_{l+k}} x^{\nu+1} q_{k-1,l+1}(x)\, dx\right. \\ &\quad \left. - \int_{t_l}^{t_{l+k-1}} x^{\nu+1} q_{k-1,l}(x)\, dx\right). \qquad \ldots (3.6)\end{aligned}$$

Now define

$$Q_{k-1,l+1}(x) = \begin{cases} q_{k-1,l+1}(x), & t_{l+1} \leqq x \leqq t_{l+k} \\ 0, & \text{otherwise} \end{cases}$$

and

$$Q_{k-1,l}(x) = \begin{cases} q_{k-1,l}(x), & t_l \leqq x \leqq t_{l+k-1} \\ 0, & \text{otherwise} \end{cases}$$

If we let

$$q_{k,l}(x) = \frac{-k}{t_{l+k}-t_l} \int_{tl}^{x} \{Q_{k-11,l+1}(t) - Q_{k-1,l}(t)\}\, dt$$

then we can write (3.6) as

$$\bar{\tau}_{\nu,k,l} = \frac{-1}{\nu+1} \int_{tl}^{t_{l+k}} x^{\nu+1}\, q'_{k,l}(x)\, dx.$$

Integrating by parts the right-hand side and simplifying, we obtain

$$\bar{\tau}_{\nu,k,l} = \int_{tl}^{t_{l+k}} x^{\nu}\, q_{k,l}(x)\, dx$$

as required, the integrated term vanishing at both the limits.

As a direct consequence of Theorem 3.3 and Lemma 3.2, we have the following important result.

Theorem 3.4—Let $t_l < t_{l+k}$ Then

$$C_{l,k}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} q_{k,l}(u)\, e^{-ixu}\, du \qquad \ldots(3.7)$$

has compact support (t_l, t_{l+k}).

Next we shall establish recursion formulae for the function $C_{k,l}(x)$.

$$\textit{Theorem } 3.5 - \frac{k-j-1}{k} C^{(j)}_{k,l}(x) = \frac{t_{l+k}-x}{t_{l+k}-t_l} C^{(j)}_{k-1,l+1}(x) + \frac{x-t_l}{t_{l+k}-t_l} C^{(j)}_{k-1,l}(x) \qquad \ldots(3.8)$$

and

$$C^{(j)}_{k,l}(x) = \frac{k}{t_{l+k}-t_l}\left\{ C^{(j-1)}_{k-1,l}(x) - C^{(j-1)}_{k-1,l+1}(x) \right\}. \qquad \ldots(3.9)$$

PROOF : From (3.2)

$$\left\langle x^\nu, \frac{1}{k} C^{(j)}_{k,l}(x) \right\rangle = \frac{(-1)^j \nu!}{k(\nu-j)!} \bar{\tau}_{\nu-j,k,l}$$

$$= \frac{(-1)^j \nu!}{k(\nu-j)!(k-j-1)} \{(\nu+k-j)\bar{\tau}_{\nu-j,k,l} - (\nu+1)\bar{\tau}_{\nu-j,k,l}\}. \qquad \ldots(3.10)$$

Now using the reperesentations (2.4) and (2.8) respectively, we have

$$(\nu+1)\bar{\tau}_{\nu-j,k,l} = \frac{\nu+1}{\nu+1-j} \cdot \frac{k}{t_{l+k}-t_l} (\bar{\tau}_{\nu,1-j,k-1,l+k} - \bar{\tau}_{\nu,1-1,k-1,l})$$

and

$$(\nu+k-j)\bar{\tau}_{\nu-j,k,l} = \frac{k}{t_{l+k}-t_l} (t_{l+k}\bar{\tau}_{\nu-j,k-1,l+1} - t_l \bar{\tau}_{\nu-j,k-1,l}).$$

Substituting the above two values in (3.10) we have

$$\left\langle x^\nu, \frac{1}{k} C^{(j)}_{k,l}(x) \right\rangle = \frac{(-1)^j \nu!}{(\nu-j)!(t_{l+k}-t_l)} \left\{ t_{l+k}\bar{\tau}_{\nu-j,k-1,l+1} - t_l \bar{\tau}_{\nu-j,k,l} - \frac{\nu+1}{\nu+1-j} (\bar{\tau}_{\nu+1-j,k-1,l+1} - \bar{\tau}_{\nu+1-j,k-1,l+1}) \right\} \qquad \ldots(3.11)$$

Once again using (3.2), we have

$$t_{l+k}\bar{\tau}_{\nu-j,k-1,l+1} - \frac{\nu+1}{\nu+1-j}\bar{\tau}_{\nu+1-j,k-1,l+1}$$

(equation continued on p. 826)

$$= t_{l+k} \frac{(k-1)(\nu-j)!}{(-1)^j \nu!} \left\langle x^\nu, \frac{1}{k-1} C^{(j)}_{k-1,l+1}(x) \right\rangle$$

$$- \frac{}{\nu+1-j} \cdot \frac{(k-1)(\nu+1-j)!}{(-1)^j(\nu+1)!}$$

$$\left\langle x^\nu, \frac{1}{k-1} C^{(j)}_{k-1,l+1}(x) \right\rangle = \frac{(k-1)!(\nu-j)!}{(-1)^j \nu!}$$

$$\left\{ \left\langle x^\nu, \frac{t_{l+k}}{k-1} C^{(j)}_{k-1,l+1}(x) \right\rangle - \left\langle x^\nu, \frac{x}{k-1} C^{(j)}_{k-1,l+1}(x) \right\rangle \right\}$$

$$= \frac{(k-1)(\nu-j!)!}{(-1)^j \nu!} \left\langle x^\nu, \frac{t_{l+k}-x}{k-1} C^{(j)}_{k-1,l+1}(x) \right\rangle \quad (3.12)$$

Similarly one can show that

$$\frac{\nu+1}{\nu+1-j} \bar{\tau}_{\nu+1-j,k-1,l} - t_l \bar{\tau}_{\nu-j,k-1,l} = \frac{(k-1)(\nu-j)!}{(-1)\nu!}$$

$$\left\langle x^\nu \frac{x-t_l}{k-1} C^{(j)}_{k-1,l}(x) \right\rangle \quad \ldots(3.13)$$

By substituting the results (3.12) and 3.13) in (3.11) we have

$$\left\langle x^\nu, \frac{1}{k} C^{(j)}_{k-l}(x) \right\rangle = \frac{k-1}{(k-j-1)(t_{l+k}-t_l} \left\langle x^\nu \frac{t_{l+k}-x}{k-1} C^{(j)}_{k-1,l+1}(x) \right.$$

$$\left. + \frac{x-t_l}{k-1} C^{(j)}_{k-1,l}(x) \right\rangle$$

when by Lemma 3.1, we obtain

$$\frac{k-j-1}{k} C^{(j)}_{k,l}(x) = \frac{t_{l+k}-x}{t_{l+k}-t_l} C^{(j)}_{k-l,l+1}(x) + \frac{x-t_l}{t_{l+k}-t_l} C^{(j)}_{k-1,l}(x)$$

proving the identity (3.8).

Once again from the identity (3.2), and using (2.4), we obtain

$$\left\langle x^\nu, C^{(j)}_{k,l}(x) \right\rangle = \frac{(-1)^j \nu! k}{(\nu-j+1)(\nu-j)!} \cdot \frac{1}{t_{l+k}-t_l} (\bar{\tau}_{\nu+1-j,\,k-1,\,l+1} - \bar{\tau}_{\nu+1-j,\,k-1,l})$$

$$= \frac{k}{t_{l+k}-t_l} \left(\left\langle x^\nu, C^{(j-1)}_{k-1,l+1}(x) \right\rangle - \left\langle x^\nu, C^{(j-1)}_{k-1,l}(x) \right\rangle \right)$$

whence the identity (3.9),

$$C^{(j)}_{k,l}(x) = \frac{k}{t_{l+k}-t_l} \left\{ C^{(j)}_{k-1,l+1}(x) - C^{(j)}_{k-1,l}(x) \right\}.$$

Corollary—$C_{k,l}(x) > 0$, $t_l < x < t_{l+k}$ and $C_{k,l}(x) = 0$. otherwise.

Theorem 3.6—If $t_l \leqq t_{l+1} \leqq \ldots \leqq t_{l+k}$ then

$$| g_{k,l}(u) | \leqq 1, \quad -\infty < u < \infty$$

and the equality holds if $t_l = t_{l+k}$ or $u = 0$.

PROOF: Clearly, using (2.2), whenever $t_l = t_{l+k'}$,

$$\left| g_{k,l}(u) \right| = \left| e^{iut_l} \right| = 1.$$

Further, if $t_l < t_{l+k}$,

$$g_{k,l}(0) = 1.$$

by definition, and

$$g_{k,l}(u) = \int_{-\infty}^{\infty} C_{k,l}(x)\, e^{iux}\, dx$$

$$= \int_{-\infty}^{\infty} C_{k,l}(x) \cos(ux)\, dx + i \int_{-\infty}^{\infty} C_{k,l}(x) \sin(ux)\, dx.$$

Therefore,

$$\left| g_{k,l}(u) \right|^2 = \left| \int_{-\infty}^{\infty} C_{k,l}(x) \cos(ux)\, dx \right|^2 + \left| \int_{-\infty}^{\infty} C_{k,l}(x) \sin(ux)\, (dx) \right|^2. \quad \ldots(3.14)$$

Now we write

$$\int_{-\infty}^{\infty} C_{k,l}(x) \cos(dx) = \int_{-\infty}^{\infty} C_{k,l}^{1/2}(x).\ C_{k,l}^{1/2}(x) \cos(ux)\, dx$$

since $C_{k,l}(x) > 0$. By Schwartz's inequality

$$\left| \int_{-\infty}^{\infty} C_{k,l}(x) \cos(ux)\, dx \right|^2 < \int_{-\infty}^{\infty} C_{k,l}(x)\, dx. \int_{-\infty}^{\infty} C_{k,l}(x) \cos^2(ux)\, dx$$

$$= \int_{-\infty}^{\infty} C_{k,l}(x) \cos^2(ux)\, dx$$

Similarly,

$$\left| \int_{-\infty}^{\infty} C_{k,l}(x) \sin(ux)\, dx \right|^2 < \int_{-\infty}^{\infty} C_{k,l}(x) \sin^2(ux)\, dx.$$

Hence from (3.14),

$$\left| g_{k,l}(u) \right|^2 < \int_{-\infty}^{\infty} C_{k,l}(x) \cos^2(ux)\, dx + \int_{-\infty}^{\infty} C_{k,l}(x) \sin^2(ux)\, dx$$

$$= \int_{-\infty}^{\infty} C_{k,l}(x)\, dx = 1.$$

as required.

Let $g_{k,l}(u) = 1$. Then, clearly from above u must be equal to zero.

It is now obvious that the function $C_{k,l}(x)$, defined as a map of $g_{k,l}(u)$ under Fourier transformation given by eqn. (3.1), is equivalent to the known k-order B-splines at the arbitrary knots $t_l \leqq t_{l+1} \leqq \ldots \leqq t_{l+k}$. Finally we shall consider a few special cases to illustrate the general nature of our results.

(i) Letting $j = 1$, in (3.9), we have

$$C_{k,l}(x) = \frac{k}{t_{l+k} - t_l} \int_{-\infty}^{x} \{ C_{k-1,l}(t) - C_{k-1,\,l+1}(t) \}\, dt$$

which gives us the usual representation of the general B-spline functions.

(ii) If $k = 1$, then

$$C_{1,l}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g_{1,l}(u)\, e^{iux}\, du$$

$$= \begin{cases} 0, & \text{if } x \geqq t_{l+1} \text{ or } x \leqq t_l \\ \dfrac{1}{t_{l+1} - t_l} & \text{if } t_l < x < t_{l+1} \end{cases}$$

And consequently one can easily evaluate $C_{k,l}(x)$, for $k = 2, 3, \ldots$, the piecewise polynomials of degree k.

(iii) The equidistant case.

Let $t_{l+j} = -\frac{1}{2}k + j$, $j = 0, 1, 2, \ldots$

Now from the definitions (2.1) and (2.2),

$$g_{1,l}(u) = \frac{1}{(t_{l+1} - t_l)\, iu} \left(e^{iut_{l+1}} - e^{iut_l} \right)$$

$$= \frac{1}{(t_{l+1} - t_l)\, iu}\, e^{iu(t_{l+1}+t_l)/2} \left(e^{iu(t_{l+1}-t_l)/2} - e^{-iu(t_{l+1}-t_l)/2} \right)$$

$$= e^{iu(t_{l+1}+t_l)/2}\, \frac{\sin u(t_{l+1} - t_l)/2}{u(t_{l+1} - t_l)/2}$$

If we take $k = 1$, then

$$t_l = -\tfrac{1}{2} \text{ and } t_{l+1} = \tfrac{1}{2},$$

and hence

$$g_{1,l}(u) = \frac{\sin u/2}{u/2}.$$

In general

$$g_{k,l}(u) = \left(\frac{\sin u/2}{u/2} \right)^k$$

whence

$$C_{k,l}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \left(\frac{\sin u/2}{u/2} \right)^k du$$

giving us the original definition of the uniform splines (Schoenberg 1946).

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