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AN ELEMENTARY EVALUATION OF A BETA TYPE INTEGRAL*

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An elementary evaluation is given for the integral of the weight function of the most general classical set of orthogonal polynomials.

1. INTRODUCTION

Ramanujan evaluated a number of important integrals, including

$$\int_{-\infty}^{\infty} |\Gamma(a+ix)\Gamma(b+ix)|^2\,dx$$

$$= \frac{\Gamma(a)\Gamma(a+\frac{1}{2})\Gamma(b)\Gamma(b+\frac{1}{2})\Gamma(a+b)\Gamma(\frac{1}{2})}{\Gamma(a+b+\frac{1}{2})}. \qquad \ldots(1.1)$$

He only claimed this when $a, b > 0$, but it also holds when $b = \bar{a}$ and $\mathrm{Re}\, a > 0$ (See Ramanujan 1915). The integrand in this integral is the weight function for an interesting set of orthogonal polynomials. [see Askey and Wilson (1982) for this orthogonality]. The most general set of orthogonal polynomials of classical type was given in Askey and Wilson (1979) when the weight function is discrete, and in Askey and Wilson (1983) when the weight function is absolutely continuous, with at most finitely many discrete mass points added. Ramanujan (1957, Chapter XVI, entry 5, p. 193) stated the sum of the weight function in the discrete case but he had been anticipated by Rogers (1895, §8, p. 29). He does not seem to have considered the integral.

There are two known evaluations of this integral, and neither one is particularly attractive. The first, in Askey and Wilson (1983), uses Cauchy's theorem to reduce the evaluation to four infinite series. These can be summed and the resulting terms put together into a single expression. However various artificial restrictions have to be made, and then removed at the end by continuity and analytic continuation. Since the final result is very attractive, this evaluation seems more complicated than necessary. The second evaluation can be derived from an asymptotic formula for some

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orthogonal polynomials due to Ismail and Wilson (1982), using results of Nevai (1979, p. 141, Corollary 36, p. 143, Theorem 40). This is also more complicated than it should be. There are special cases that can be evaluated easily, and it is now possible to reduce the general case to one of these special cases. This will be done in the next section.

2. The Integral

Let a, b, c, d and q be complex numbers, $|q| < 1$, $\max(|a|, |b|, |c|, |d|) < 1$, with the exception that one of a, b, c, d can be one, and one can be minus one. Define

$$h(x, a) := \prod_{n=0}^{\infty} (1 - 2axq^n + a^2q^{2n}).$$

Consider the integral

$$I(a, b, c, d) = \frac{1}{2\pi} \int_{-1}^{1} \frac{h(x, 1)\, h(x, q^{1/2})\, h(x, -1)\, h(x, -q^{1/2})\, dx}{h(x, a)\, h(x, b)\, h(x, c)\, h(x, d)\, (1 - x^2)^{1/2}}. \qquad \ldots(2.1)$$

Either square root can be taken for $q^{1/2}$, since both occur in a symmetric fashion. The key to the evaluation of (2.1) is

$$bI(aq, b, c, d) = aI(a, bq, c, d) + (b - a)(1 - ab)\, I(a, b, c, d). \qquad \ldots(2.2)$$

This holds because

$$b(1 - 2ax + a^2) = a(1 - 2bx + b^2) + (b - a)(1 - ab).$$

To evaluate (2.1) we start with a special case. Take $b = 1$, $c = -1$, $d = -q^{1/2}$. Then if

$$I(a) = \frac{1}{2\pi} \int_{-1}^{1} \frac{h(x, q^{1/2})\, dx}{h(x, a) \sqrt{1 - x^2}}$$

we take $x = \cos\theta$ to obtain

$$I(a) = \frac{1}{2\pi} \int_{0}^{\pi} \frac{(q^{1/2}e^{i\theta}; q)_\infty\, (q^{1/2}e^{-i\theta}; q)_\infty\, d\theta}{(ae^{i\theta}; q)_\infty\, (ae^{-i\theta}; q)_\infty}. \qquad \ldots(2.3)$$

Here

$$(a; q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n). \qquad \ldots(2.4)$$

Define

$$(a; q)_n = (a; q)_\infty / (aq^n; q)_\infty. \qquad \ldots(2.5)$$

Ramanujan [1957, Chapter XVI, entry 17, p. 196] stated that

$$\frac{(\beta/\alpha; q)_\infty (q; q)_\infty (\alpha t; q)_\infty (q/\alpha t; q)_\infty}{(q/\alpha; q)_\infty (\beta; q)_\infty (t; q)_\infty (\beta/\alpha t; q)_\infty} = \sum_{-\infty}^{\infty} \frac{(\alpha; q)_n}{(\beta; q)_n} t^n.$$

Simple proofs are given in Ismail (1977) and Askey (1980). Take $t = ae^{i\theta}$, $\alpha = q^{1/2}a^{-1}$, $\beta = q^{1/2}a$ to get

$$I(a) = \frac{1}{2\pi} \frac{(aq^{1/2}; q)_\infty^2}{(q; q)_\infty (a^2; q)_\infty} \sum_{-\infty}^{\infty} \frac{(q^{1/2}a^{-1}; q)_n}{(q^{1/2}a; q)_n} a^n \int_0^\pi e^{in\theta}\, d\theta.$$

This can be changed in a simple way to have the integral over $[-\pi, \pi]$, for the integrand in (2.3) is an even function of θ. Thus

$$I(a) = \frac{(aq^{1/2}; q)_\infty^2}{2(q; q)_\infty (a^2; q)_\infty}. \qquad \ldots(2.6)$$

Now we claim that

$$I(a, b, c, d) = \frac{(abcd; q)_\infty}{(q; q)_\infty (ab; q)_\infty (ac; q)_\infty (ad; q)_\infty (bc; q)_\infty (bd; q)_\infty (cd; q)_\infty}. \qquad \ldots(2.7)$$

When $b = 1$, $c = -1$, $d = -q^{1/2}$ this reduces to

$$\frac{(aq^{1/2}; q)_\infty}{(q; q)_\infty (a; q)_\infty (-a; q)_\infty (-aq^{1/2}; q)_\infty (-1; q)_\infty (-q^{1/2}; q)_\infty (q^{1/2}; q)_\infty}$$

$$= \frac{(aq^{1/2}; q)_\infty^2}{2(q; q)_\infty (a^2; q^2)_\infty (a^2q; q^2)_\infty (-q; q)_\infty (q; q^2)_\infty}$$

$$= \frac{(aq^{1/2}; q)_\infty^2 (q; q)_\infty}{2(q; q)_\infty (a^2; q)_\infty (q^2; q^2)_\infty (q; q^2)_\infty} = I(a).$$

Thus (2.7) is true when $b = -1$, $c = -1$, $d = -q^{1/2}$. Assume that (2.7) has been proven for all a when b, c, d are fixed. Then it is true for all a when b is replaced by bq, for the right-hand side of (2.7) and $I(a, b, c, d)$ both satisfy (2.2). Thus (2.7) is true for $(a, q^k, -1, -q^{1/2})$. But $I(a, b, c, d)$ and the right-hand side of (2.7) are both analytic functions of the parameters a, b, c, d when $\max(|a|, |b|, |c|, |d|) < 1$, and q^k is an infinite sequence with a limit point in this set, so (2.7) holds for $(a, b, -1, -q^{1/2})$. By the same argument it holds for $(a, b, -q^k, -q^{1/2})$ and $(a, b, c, -q^{k+1/2})$,

and so for (a, b, c, d) when $\max(|a|, |b|, |c|, |d|) < 1$. By the proof it also holds when $b = 1$ and/or $c = -1$.

The above argument is not really an evaluation of this integral, but a verification that the assumed value is correct. It would still be interesting to have a simple evaluation. It is a shame Ramanujan was not aware of this integral, for he would have liked it.

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