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WEYL'S INEQUALITY AND SIMULTANEOUS ADDITIVE EQUATIONS

R. J. COOK

University of Sheffield, Hounsfield Road, Sheffield S3 7RH

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We obtain a version of Weyl's inequality suitable for R additive equations of degree k and consider particular applications for several small values of R and k.

1. INTRODUCTION

In this memoir, we continue our researches concerning the solubility of simultaneous additive equations

$$a_{i1} x_1^k + \ldots + a_{iN} x_N^k = 0, \quad i = 1,\ldots,R \qquad \ldots(1)$$

with integer coefficients a_{ij}. The number of integer solutions of (1) lying in a box

$$c_j P_j < x_j < C_j P_j \qquad \ldots(2)$$

is given by the integral

$$\int_0^1 \ldots \int_0^1 \prod_{j=1}^{N} T_j(\Lambda_j)\, d\alpha_1 \ldots d\alpha_R \qquad \ldots(3)$$

where

$$\Lambda_j = \sum_{i=1}^{R} a_{ij}\, \alpha_i, \quad j = 1,\ldots,N, \qquad \ldots(4)$$

$$T_j(\Lambda_j) = \sum_{(2)} e(\Lambda_i x_j^k) \qquad \ldots(5)$$

and $e(\theta) = \exp(2\pi i\theta)$.

In evaluating the integral (3) by the circle method it is normal to take the major arcs $M(\mathbf{A}, Q)$ to consist of those α in the unit cube having rational approximations

$$\left|\alpha_i - \frac{A_i}{Q}\right| < Q^{-1} P^{-k+1-\delta}, \quad i = 1,\ldots,R, \qquad \ldots(6)$$

where δ is some small positive constant, $(A_1,\ldots,A_R, Q) = 1$ and

$$1 \leqslant Q \leqslant P^{1-\delta}. \qquad \ldots(7)$$

In order to estimate the contribution of the minor arcs $\mathfrak{m}$ (i.e. the rest of the unit cube) we need some version of Weyl's estimate. Rather surprisingly, the following version of Weyl's estimate appears to be new, although the result must have been known in principle to Davenport and Lewis, [see Lemma 19 of Davenport and Lewis (1969)].

Theorem 1—Let $\Lambda_1,\ldots, \Lambda_R$ be independent linear forms in $\alpha_1,\ldots,\alpha_R$ with integer coefficients. If $\vec{\alpha} \in \mathfrak{m}$ then

$$\min_{j=1,\cdots,R} | T_j(\Lambda_j) | \ll P^{1-\tau+\delta} \qquad \ldots(8)$$

where

$$\tau = \min((2^{k-1})^{-1}, (Rk)^{-1}). \qquad \ldots(9)$$

In the later sections of the paper, we apply this to particular cases when R and k are both small. Our motivation comes from Artin's conjecture which suggests that $Rk^2 + 1$ variables should be sufficient. An application of Theorem 1 with our version of Hua's Lemma, [see Cook (1972a)], shows that $R2^k+1$ variables are sufficient for the analytic arguments provided that the columns of the matrix (a_{ij}) are in sufficiently general position. Specifically, the matrix (a_{ij}) should contain $R + 1$ columns in general position and a further $(2^k - 1)$ sets of R independent columns.

Whilst this shows that $Rk^2 + 1$ variables are sufficient for the analytic arguments when $k = 2$, 3 or 4, in some cases better results are possible. In the case $k = 3$, $R = 2$, Devenport and Lewis (1966) showed that 16 variables are sufficient for the p-adic arguments and that 18 variables are sufficient for the analytic arguments. Subsequently the '18' was reduced to 17 by Cook (1972b) and 16 by Vaughan (1977).

2. Proof of Theorem 1

Let

$$| T_j(\Lambda_j) | = P^{1-\theta_j}, \quad j = 1,\ldots,R \qquad \ldots(10)$$

and suppose that

$$\theta_j \leqslant \tau - \delta, \quad j = 1,\ldots, R. \qquad \ldots(11)$$

For each linear from Λ_j we can choose integers a_j, q_j satisfying

$$| q_j \Lambda_j - a_j | < P^{-k+1-\delta}, \ 1 \leqslant q_j \leqslant P^{k-1+\delta}, \ (a_j, q_j) = 1. \qquad \ldots(12)$$

If $q_j \geqslant P^{1-\delta}$ then it follows from Weyl's inequality, [see Lemma 1 of Davenport (1962)], that

$$| T_j(\Lambda_j) | \ll P^{1-\kappa+\delta} \qquad \ldots(13)$$

where $\kappa = 2^{1-k}$. Now we may suppose that $q_j \leqslant P^{1-\delta}$ for $j = 1,\ldots,R$ and will show that (11) implies that $\vec{\alpha} \notin \mathfrak{m}$.

Let

$$\Lambda_j = a_j/q_j + \gamma_j \qquad \ldots(14)$$

then, from Lemma 18 of Davenport and Lewis (1969), we have

$$P^{1-\theta_j} = |T_j(\Lambda_j)| \ll q_j^{-1/k} \min(P, P^{1-k}|\gamma_j|^{-1}). \qquad \ldots(15)$$

Therefore

$$q_j \ll P^{k\theta_j}, \quad j = 1,\ldots,R, \qquad \ldots(16)$$

and

$$|\gamma_j| \ll q_j^{-1/k} P^{-k+\theta_j}, \; j = 1, \ldots,R. \qquad \ldots(17)$$

Let

$$\Lambda_j = \sum_{i=1}^{R} c_{ij}\alpha_i, \quad j = 1,\ldots,R, \qquad \ldots(18)$$

where the coefficients c_{ij} are integers with $\Delta = \det c_{ij} \neq 0$. Solving these equations for $\alpha_1,\ldots,\alpha_R$ by Cramer's rule we find that

$$\Delta\alpha_i = \sum_{j=1}^{R} d_{ij}\Lambda_j$$

$$= \sum_{j=1}^{R} d_{ij}\left[\frac{a_j}{q_j} + \gamma_j\right], \; i = 1,\ldots,R, \qquad \ldots(19)$$

where the d_{ij} are the cofactors of (c_{ij}). Thus

$$\alpha_i = A_i/Q + \beta_i \qquad \ldots(20)$$

where

$$\beta_i = \Delta^{-1}\sum_{j=1}^{R} d_{ij}\gamma_j \qquad \ldots(21)$$

and $A_1,\ldots,A_R$, Q are integers with $(A_1,\ldots,A_R, Q) = 1$ and

$$Q \mid \Delta q_1\ldots q_R$$

so that

$$Q \leqslant |\Delta| q_1\ldots q_R \ll P^{k(\theta_1+\ldots+\theta_R)}. \qquad \ldots(22)$$

Now $\theta_j \leqslant \tau - \delta$ and $\tau \leqslant 1/Rk$, so that

$$k(\theta_1 + \ldots + \theta_R) < 1 - \delta. \qquad \ldots(23)$$

Therefore, if P is large enough, $Q \leqslant P^{1-\varepsilon}$. Also

$$Q \mid \beta_i \mid \leqslant q_1 \ldots q_R \sum_{j=1}^{R} \mid d_{ij} \parallel \gamma_j \mid$$

$$\ll \sum_{j=1}^{R} q_1 \ldots q_R \, q_j^{-1/k} \, P^{-k+\theta_j}$$

or

$$\ll P^{k(\theta_1 + \cdots + \theta_R) - k}$$

by (16) and (17). From (23), we see that if P is large enough

$$Q \mid \beta_i \mid \leqslant P^{-k+1-\varepsilon} \qquad \ldots(24)$$

and so, as $Q \leqslant P^{1-\varepsilon}$, $\vec{\alpha}$ is in the major arcs. This contradiction shows that at least one of the T_j's satisfies

$$\mid T_j(\Lambda_j) \mid \ll P^{1-r+\varepsilon}.$$

3. Admissible Exponents and the Minor Arcs

Recently (Cook 1983), we have extended results of Davenport (1939) on admissible exponents to simultaneous additive equations.

Definition—The s real numbers $\lambda_1, \ldots, \lambda_s$ satisfying

$$\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_s > 0 \qquad \ldots(25)$$

are called admissible exponents (for R additive equations of degree k), if, for any $R \times sR$ integer matrix A whose columns form s consecutive non-singular $R \times R$ matrices, the number of solutions of

$$AX = AY \qquad \ldots(26)$$

where

$$X = \left[x_1^k, \ldots, x_{sR}^k \right]^T, \; Y = \left[y_1^k, \ldots, y_{SR}^k \right]^T \qquad \ldots(27)$$

are integer vectors satisfying

$$c_j P^{\lambda_\nu} < x_j, \quad y_j < C_j P^{\lambda_\nu} \qquad \ldots(28)$$

with $\nu = 1 + [(j-1)/R]$ and $0 < c_j < C_j$, is

$$O\left(P^{R(\lambda_1 + \cdots + \lambda_s) + \epsilon}\right) \text{ as } P \to \infty \qquad \ldots(29)$$

for any $\epsilon > 0$.

Lemma 1—Let $\lambda_1, \ldots, \lambda_s$ be admissible exponents for R additive equations of degree k. Let

$$\sigma = \lambda_1 + \ldots + \lambda_s \text{ and } \Delta = k\lambda_1 - (k-1). \qquad \ldots(30)$$

If $\Delta > 0$, then $1, \lambda_1, \ldots, \lambda_s$ are also admissible exponents provided that there exists an integer satisfying

$$1 \leqslant r \leqslant k - 2 \qquad \ldots(31)$$

$$2^r \Delta \leqslant 1 \qquad \ldots(32)$$

$$(2^r - 1)\Delta + \sigma \leqslant r + 1. \qquad \ldots(33)$$

This is Theorem 3 of Cook (1983); it is a generalization of Theorem 2 of Davenport (1939).

We apply Lemma 1 with s admissible exponents $\lambda_1 = 1, \lambda_2, \ldots, \lambda_s$ using $2s$ blocks of R variables, for which the corresponding suffices j lie in blocks $\mathscr{B}_1, \ldots, \mathscr{B}_s, \ldots, \mathscr{B}_1', \ldots, \mathscr{B}_s'$. Let

$$N = t + 2Rs \qquad \ldots(34)$$

where the extra t variables x_j, say $j \in \mathscr{C}$, run through ranges of length $\geqslant P$.

We consider the variables x_j with $j \in \mathscr{C} \cup \mathscr{B}_1$. Suppose that the columns of the corresponding submatrix of coefficients are in general position. For each fixed $\vec{\alpha} \in \mathbf{m}$ we apply Theorem 1 repeatedly to obtain a saving of $P^{r-\delta}$ for each of t exponential sums T_j, say $j \in \mathscr{T}$, and are left with R variables x_j for which the suffices form a new block $\mathscr{B}^*$. Let $\mathbf{m}^*$ denote the subset of $\mathbf{m}$ consisting of all those $\vec{\alpha} \in \mathbf{m}$ corresponding to the same blocks $\mathscr{B}^*$ and $\mathscr{T}$. Then

$$\max_{\vec{\alpha} \in \mathbf{m}^*} \prod_{j \in \mathscr{T}} |T_j(\Lambda_j)| \ll P^{t(1-r+\delta)}. \qquad \ldots(35)$$

The contribution of the variables x_j with $j \in \mathscr{B}^*, \mathscr{B}_2, \ldots, \mathscr{B}_s'$ is estimated using Lemma 1,

$$\int_0^1 \int_0^1 \prod |T_j(\Lambda_j)| \, d\vec{\alpha} \ll P^{R(\lambda_1 + \ldots + \lambda_s) + \epsilon}. \qquad \ldots(36)$$

Summing over $\binom{R+t}{R}$ possible blocks $\mathscr{T}^*$ we obtain a saving in the exponent of

$$t(\tau - \delta) + R(\lambda_1 + \ldots + \lambda_s) - \epsilon.$$

Let $\sigma = \lambda_1 + \ldots + \lambda_s$, if

$$t\tau + R\sigma > Rk \qquad \ldots(37)$$

then these estimates will be adequate for the treatment of the minor arcs.

Before proceeding to the treatment of the major arcs we recall our requirements on the coefficient matrix; there are $t + R$ columns in general position and the remaining $(2s - 1)R$ columns may be partitioned into $(2s - 1)$ blocks of R independent columns.

4. The Major Arcs and the Singular Series

On the major arc $M(\mathbf{A}, Q)$ we have

$$\Lambda_j = \sum_{i=1}^{R} a_{ij}\,\alpha_i = \sum_{i=1}^{R} a_{ij}\left[\frac{A_i}{Q} + \beta_i\right]$$

$$= \frac{a_j}{q_j} + \gamma_j \qquad \ldots(38)$$

where $(a_j, q_j) = 1$

$$\frac{a_j}{q_j} = \frac{1}{Q}\sum_{i=1}^{R} a_{ij}\,A_i \qquad \ldots(39)$$

and

$$\gamma_j = \sum_{i=1}^{R} a_{ij}\,\beta_i. \qquad \ldots(40)$$

Let $\mathscr{H} = \mathscr{B}_1 \cup \mathscr{B}_2 \cup \mathscr{C}$, then $t+2R$ suffices in $\mathscr{H}$ correspond to the exponential sums of length $\geqslant P$. For $j \in \mathscr{H}$ and $\vec{\alpha} \in \mathbf{M}$ we have

$$|T_j(\Lambda_j)| \ll q_j^{-1/k} \min(P, P^{1-k}|\gamma_j|^{-1}) \qquad \ldots(41)$$

[see Lemma 18 of Davenport and Lewis (1969)].

Lemma 2—We have

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \prod_{j\in\mathscr{H}} \min(P, P^{1-k}|\gamma_j|^{-1})\,d\vec{\gamma} \ll P^{t+2R-Rk} \qquad \ldots(42)$$

and for any $\eta > 0$

$$\int_{D(\eta)}\cdots\int \prod_{j\in\mathscr{H}} \min(P, P^{1-k}|\gamma_j|^{-1})\,d\vec{\gamma} \ll P^{t+2R-Rk-\eta} \qquad \ldots(43)$$

where

$$D(\eta) = \{\vec{\gamma} : \max |\gamma_j| > P^{-k+\eta}\}. \qquad \ldots(44)$$

This may be proved in the same way as Lemma 22 of Davenport and Lewis (1969).

The treatment of the singular series used by Davenport and Lewis (1969) uses only the exponential sums over a full range $\geqslant P$. It is preferable to include some of the short range exponential sums in the singular series. Let

$$\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_u \qquad \ldots(45)$$

be those admissible exponents λ for which

$$\lambda\,(k-1) > 1. \qquad \ldots(46)$$

Then, as in Lemma 34 of Davenport and Lewis (1966), we have

$$|\, T_j\,(\Lambda_j)\,| \ll q_j^{-\lambda\kappa}\, P^{\lambda+\delta}\ (\kappa = 2^{1-k}) \qquad \ldots(47)$$

for these short-range exponential sums.

Let

$$\Omega = t + 2R\sigma\ (\sigma = \lambda_1 + \ldots + \lambda_s). \qquad \ldots(48)$$

Then on the major arcs we have

$$\prod_{j=1}^{N} |\, T_j\,(\Lambda_j)\,| \ll P^{\alpha+N\delta-(t+R)} \prod_{j\in\mathscr{H}} q_j^{-1/k}\, \min\,(P,\ P^{1-k}\,|\,\gamma_j\,|^{-1})$$

$$\times \prod_{i=2}^{u} \prod_{j=1}^{2R} q_{i,j}^{-\lambda_i\kappa} \qquad \ldots(49)$$

where $q_{i,j}$ refers to the jth linear form in the blocks $\mathscr{B}_i$, $\mathscr{B}'_i$ corresponding to the ith admissible exponent.

The singular series will develop from the sum

$$\sum_{Q}\sum_{A} \prod_{j\in\mathscr{H}} q_j^{-1/k} \cdot \prod_{i=2}^{u} \prod_{j=1}^{2R} q_{i,j}^{-\lambda_i\kappa} \qquad \ldots(50)$$

where the sum is over those values of Q, $\mathbf{A}$ occurring on the major arcs. Let

$$\phi = \lambda_2 + \ldots + \lambda_u \qquad \ldots(51)$$

and

$$\theta = \frac{t+2R}{Rk} + 2\,\phi\kappa. \qquad \ldots(52)$$

Using Hölder's inequality the sum in (50) can be majorised by sums

$$\sum_{Q}\sum_{A} (q_1 \ldots q_R)^{-\theta} \qquad \ldots(53)$$

where the product $q_1 \ldots q_R$ corresponds to independent linear forms. Following Davenport and Lewis, Lemma 23 of the 1969 paper and Lemma 35 of the 1966 paper, the singular series will converge absolutely provided that

$$\theta > 2\ (R \geqslant 3) \text{ or } \theta > 3/2\ \ (R = 2). \qquad \ldots(54)$$

Using this, Lemma 2 and following § 11 of Davenport and Lewis (1969), the major arcs can then be contracted to

$$\mathfrak{M}_0(A, Q) : \max |\gamma_i| < P^{-k+\omega}, Q \leqslant P^{\omega} \quad \ldots(55)$$

where ω is a small positive constant. On the contracted major arcs there is a good approximation to the exponential sum $T_j(\Lambda_j)$, see Lemma 27 of Davenport and Lewis (1969), and this leads to the asymptotic formula

$$P^{a-Rk}(\mathfrak{S} + o(1))(\mathfrak{J} + o(1)) \quad \ldots(56)$$

for the integral (3).

Here $\mathfrak{S}$ is the singular series, which converges absolutely if (54) is satisfied, and which will then be positive provided that the equations (1) have a non-singular solution in every p-adic field. The term $\mathfrak{J}$ arises from the singular integral, we have $\mathfrak{J} > 0$ provided that

$$t + 2R > Rk \quad \ldots(57)$$

and the eqns. (1) have a non-singular real solution satisfying

$$c_j P < x_j < C_j P \quad \text{for } j = 1, \ldots, t + 2R$$

and $x_j = 0$ for $j > t + 2R$. Since we shall suppose that the $t + R$ columns of the matrix $a_{ij} \in \mathscr{B} \cup \mathscr{C}$, are in general position it follows from §7 of Davenport and Lewis (1969) that we need only assume that eqns. (1) have a non-singular solution. This is clearly satisfied automatically when k is odd.

5. Quartic Equations

When $R = 2$ it follows from the p-adic results of Davenport and Lewis (1967) and the analytical methods of Cook (1972b) that the equations

$$\begin{aligned} f(\mathbf{x}) &= a_1 x_1^4 + \ldots + a_N x_N^4 = 0 \\ g(\mathbf{x}) &= b_1 x_1^4 + \ldots + b_N x_N^4 = 0 \end{aligned} \quad \ldots(58)$$

will have a non-trivial solution in integers provided that

(i) $N \geqslant 33$;

(ii) every member of the pencil $\lambda f + \mu g$, $(\lambda, \mu) \neq (0, 0)$, contains at least 17 variables explicitly; and

(iii) the equations have non-trivial real and 2-adic solutions.

However, if we assume the existence of p-adic solutions then 28 variables are sufficient. We take $t = 8$ and $s = 5$, iterating from the admissible exponents 1, 1 we have admissible exponents

$$1, \frac{72}{91}, \frac{9}{14}, \frac{15}{28}, \frac{15}{28}, \qquad \ldots(59)$$

[see Theorem 5 of Davenport (1939)]. Now

$$\tau = 1/8 \text{ and } \sigma = 319/91 \qquad \ldots(60)$$

so

$$t\tau + R\sigma = 8 + 1/91 > 8 \qquad \ldots(61)$$

and (37) is satisfied. Also $t + 2R = 12 > 8$ and taking $u = 5$

$$\frac{t + 2R}{Rk} + 2(\sigma - 1)\kappa = \frac{3}{2} + \frac{1}{8}\left[5 + \frac{1}{91}\right] > \frac{3}{2} \qquad \ldots(62)$$

so (54) and (57) are satisfied.

Theorem 2—The equations

$$\begin{aligned} f(x) &= a_1 x_1^4 + \ldots + a_N x_N^4 = 0 \\ g(x) &= b_1 x_1^4 + \ldots + b_N x_N^4 = 0 \end{aligned} \qquad \ldots(63)$$

will have a non-trivial solution in integers provided that

(i) $N \geqslant 28$;

(ii) the ratios a_i/b_i contain a set $\mathscr{S}$ of 10 distinct ratios and the remaining ratios contain a set $\mathscr{T}$ of 18 ratios which can be partitioned into 9 pairs of distinct ratios; and

(iii) the equations have non-singular real and p-adic solutions.

Applying Lemma 1, with $l = 2$, to the admissible exponents in (59) we obtain a sequence of 6 addmissible exponents

$$\lambda_1 = 1, \lambda_2 = \frac{1092}{1411}, \lambda_3 = \frac{1092}{1411} \cdot \frac{72}{91} = \frac{864}{1411},$$

$$\lambda_4 = \frac{1092}{1411} \cdot \frac{9}{14} = \frac{702}{1411}, \lambda_5 = \lambda_6 = \frac{1092}{1411} \cdot \frac{15}{28} = \frac{585}{1411} > \frac{1}{3}.$$

Then

$$\sigma(6) = \lambda_1 + \ldots + \lambda_6 = 1 + \frac{1092}{1411} \cdot \frac{319}{91} = \frac{5239}{1411}. \qquad \ldots(64)$$

For $R = 3$, $\tau = 1/12$ so taking $t = 11$ we have

$$t\tau + R\sigma = \frac{11}{12} + \frac{15717}{1411} = 12 + \frac{941}{16932} > 12. \qquad \ldots(65)$$

Also, $t + 2R = 17 > Rk$ and taking $u = 6$

$$\frac{t+2R}{Rk} + 2(\sigma - 1)\kappa = \frac{17}{12} + \frac{1}{4}\cdot\frac{3828}{1411} > 2 \qquad \ldots(66)$$

so (54) and (57) are both satisfied.

Theorem 3—Let $k = 4$ and $R = 3$, then the equations (1) will have a non-trivial solution in integers provided that

(i) $N \geqslant 47$;

(ii) the equations have non-singular real and p-adic equations; and

(iii) the columns of the matrix (a_{ij}) contain a set $\mathscr{S}$ of 14 columns in general position and a further set $\mathscr{T}$ of 33 colmuns which can be rearranged to form 11 non-singular 3×3 matrices.

Since the conditions on the rank of the submatrices of (a_{ij}) appear restrictive it is perhaps worth pointing out that if we consider the $T^{RN}(1 + o(1))$, matrices a_{ij} whose elements satisfy $|a_{ij}| \leqslant T/2$ then almost all, i. e. $T^{RN}(1 + o(1))$ of these matrices will satisfy the rank conditions.

6. Quintic Equations

The case $R = 2$ has already been discussed fully in Cook (1983); we showed there that when $k = 5$ there is a sequence of 9 admissible exponents with $\lambda_1 = 1$,

$$\sigma = \lambda_1 + \ldots + \lambda_9 = 4.663883\ldots \qquad \ldots(67)$$

$$\lambda_7 = 0.263514\ldots > 1/4 \qquad \ldots(68)$$

and

$$\phi = \lambda_2 + \ldots + \lambda_7 = 3.333478\ldots . \qquad \ldots(69)$$

When $R = 3$ we have $\tau = 1/16$. Taking $s = 9$, $t = 18$ and $u = 7$ we have

$$t\tau + 3\sigma = 15.116649\ldots > 15 \qquad \ldots(70)$$

$t + 2R = 24 > 15$ and

$$\frac{t+2R}{Rk} + 2\phi\kappa = \frac{24}{15} + \frac{3.333748}{8} = 2.016\ldots > 2. \qquad \ldots(71)$$

Theorem 4—Let $k = 5$ and $R = 3$, then eqns. (1) will have a non-trivial solution in integers $x_1 \ldots x_N$ if

(i) $N \geqslant 72$:

(ii) the equations have non-singular solutions in every p-adic field; and

(iii) the matrix (a_{ij}) contains a set $\mathscr{S}$ of 21 columns in general position and a further set $\mathscr{T}$ of 51 columns which can be rearranged to form 17 non-singular 3×3 matrices.

When $R = 4$, $\tau = 1/20$ and taking $s = 9$, $t = 27$ and $u = 7$ we have

$$t\tau + 4\sigma = 20.005532\ldots > 20 \qquad \ldots(72)$$

$t + 2R = 35 > 20$ and

$$\frac{t + 2R}{Rk} + 2\phi_K = \frac{35}{20} + \frac{3.333478}{8}\ldots > 2. \qquad \ldots(73)$$

Theorem 5—Let $k = 5$ and $R = 4$, then eqns. (1) will have a non-trivial solution in integers $x_1,\ldots,x_N$ if

(i) $N \geqslant 99$

(ii) the equations have non-singular solutions in every p-adic field; and

(iii) the matrix (a_{ij}) contains a set $\mathscr{S}$ of 31 columns in general position and a further set $\mathscr{T}$ of 68 columns which can be rearranged to form 17 non-singular 4×4 matrices.

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