*Indian J. pure appl. Math.*, **14**(2) : 175–178, February 1983

# A NOTE ON THE EXPANSION OF CYLINDRICAL EIGENVECTORS IN TERMS OF SPHERICAL EIGENVECTORS

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(*Received* 25 *February* 1982)

The object of this note is to obtain expansions of the vector solutions of the dynamical Navier equation in cylindrical coordinates in terms of its solutions in spherical coordinates. These expansions are useful in wave propagation problems because they immediately give the representation of the source or the wave field in spherical coordinates if the corresponding representation in cylindrical coordinates is known.

The Navier equation of dynamical elasticity may be written in the form

$$\alpha^2 \text{ grad div } \mathbf{u} - \beta^2 \text{ curl curl } \mathbf{u} + \omega^2 \mathbf{u} = 0 \qquad \ldots(1)$$

where $\mathbf{u}$ is the displacement vector, $\omega$ the circular frequency and $\alpha$, $\beta$ the compressional and shear wave velocities, respectively. Three independent solutions of (1) in cylindrical coordinates $(r, \phi, z)$ are (Ben-Menahem and Singh 1981, Ch. 2)

$$\mathbf{L}_m^{\pm} = \frac{1}{k_\alpha} \text{ grad } \Phi_m^{\pm}$$

$$= \frac{1}{k_\alpha} (\pm a \mathbf{P}_m + k \mathbf{B}_m) \exp(\pm az) \qquad \ldots(2)$$

$$\mathbf{M}_m^{\pm} = \frac{1}{k} \text{ curl } \left( \mathbf{e}_z \Psi_m^{\pm} \right)$$

$$= \mathbf{C}_m \exp(\pm bz) \qquad \ldots(3)$$

$$\mathbf{N}_m^{\pm} = \frac{1}{k k_\beta} \text{ curl curl } \left( \mathbf{e}_z \Psi_m^{\pm} \right)$$

$$= \frac{1}{k_\beta} (k\mathbf{P}_m \pm b \mathbf{B}_m) \exp(\pm bz) \qquad \ldots(4)$$

where

$$k_\alpha = \omega/\alpha, \; k_\beta = \omega/\beta$$

$$a^2 = k^2 - k_\alpha^2, \; b^2 = k^2 - k_\beta^2$$

$$\Phi_m^{\pm} = Y_m \exp(\pm az)$$

$$\Psi_m^{\pm} = Y_m \exp(\pm bz)$$

$$\begin{aligned} Y_m &= j_m(kr) \exp(im\phi) \\ \mathbf{P}_m &= \mathbf{e}_z Y_m \\ \mathbf{B}_m &= \frac{1}{k} \text{ grad } Y_m \\ \mathbf{C}_m &= \frac{1}{k} \text{ curl } (\mathbf{e}_z Y_m). \end{aligned} \quad \ldots(5)$$

Similarly, three independent solutions of (1) in spherical coordinates $(R, \theta, \phi)$ are

$$\begin{aligned} \mathbf{L}_{ml} &= \frac{1}{k_\alpha} \text{ grad } \Phi_{ml} \\ &= \mathbf{P}_{ml}\, j_l'(k_\alpha R) + \sqrt{l(l+1)}\, \mathbf{B}_{ml} j_l(k_\alpha R)/(k_\alpha R) \end{aligned} \quad \ldots(6)$$

$$\begin{aligned} \mathbf{M}_{ml} &= \text{curl } (\mathbf{R}\, \Psi_{ml}) \\ &= \sqrt{l(l+1)}\, \mathbf{C}_{ml} j_l(k_\beta R) \end{aligned} \quad \ldots(7)$$

$$\begin{aligned} \mathbf{N}_{ml} &= \frac{1}{k_\beta} \text{ curl curl } (\mathbf{R}\Psi_{ml}) \\ &= \mathbf{P}_{ml}\, l(l+1)\, j_l(k_\beta R)/(k_\beta R) \\ &\quad + \sqrt{l(l+1)}\, \mathbf{B}_{ml} \left[ j_l'(k_\beta R) + j_l(k_\beta R)/(k_\beta R) \right] \end{aligned} \quad \ldots(8)$$

where $j_l(x)$ is the spherical Bessel function, a prime denotes differentiation with respect to the argument and

$$\begin{aligned} R^2 &= r^2 + z^2, \ \mathbf{R} = R\, \mathbf{e}_R \\ \Phi_{ml} &= j_l(k_\alpha R)\, Y_{ml} \\ \Psi_{ml} &= j_l(k_\beta R)\, Y_{ml} \\ Y_{ml} &= P_l^m(\cos\theta) \exp(im\phi) \\ \mathbf{P}_{ml} &= \mathbf{e}_R\, Y_{ml} \\ \sqrt{l(l+1)}\, \mathbf{B}_{ml} &= R \text{ grad } Y_{ml} \\ \sqrt{l(l+1)}\, \mathbf{C}_{ml} &= \text{curl } (\mathbf{R} Y_{ml}). \end{aligned} \quad \ldots(9)$$

In wave propagation problems it is sometimes required to express the cylindrical vector solutions $\mathbf{L}_m$, $\mathbf{M}_m$, $\mathbf{N}_m$ in terms of the spherical vector solutions $\mathbf{L}_{ml}$, $\mathbf{M}_{ml}$, $\mathbf{N}_{ml}$, and vice-versa. The expansions of the spherical vector solutions in terms of the cylindrical vector solutions are already known (Ben-Menahem and Singh 1981, p.78). Here, our aim is to express the cylindrical vector solutions in terms of the spherical vector solutions. To this end, we note that the cylindrical wave functions $\Phi_m^{\pm}$ can be expressed in terms of the spherical wave functions $\Phi_{ml}$ (Ben-Menahem and Singh 1981, p.77)

$$\Phi_m^{\pm} = \sum_{l=m}^{\infty} i^{m-l}\,(2l+1)\,\frac{(l-m)!}{(l+m)!}\;P_l^m\left(\pm\frac{ia}{k_\alpha}\right)\Phi_{ml}. \qquad \ldots(10)$$

The functions $\Psi_m^{\pm}$ have a similar expansion in terms of $\Psi_{ml}$ with a replaced by $b$. Equations (2), (6) and (10) yield

$$\begin{aligned}\mathbf{L}_m^{\pm} &= \frac{1}{k_\alpha}\ \text{grad}\ \Phi_m^{\pm}\\ &= \frac{1}{k_\alpha}\ \text{grad}\left[\sum_{l=m}^{\infty} i^{m-l}\,(2l+1)\frac{(l-m)!}{(l+m)!}\,P_l^m\left(\pm\frac{ia}{k_\alpha}\right)\Phi_{ml}\right]\\ &= \sum_{l=m}^{\infty} i^{m-l}\,(2l+1)\,\frac{(l-m)!}{(l+m)!}\;P_l^m\left(\pm\frac{ia}{k_\alpha}\right)\mathbf{L}_{ml}. \qquad \ldots(11)\end{aligned}$$

Equation (11) expresses the cylindrical vectors $\mathbf{L}_m^{\pm}$ in terms of the spherical vectors $\mathbf{L}_{ml}$. To find the corresponding expansion for $\mathbf{M}_m$, we note from (3) and (10) that

$$\begin{aligned}\mathbf{M}_m^{\pm} &= \frac{1}{k}\ \text{curl}\left(\mathbf{e}_z\,\Psi_m^{\pm}\right)\\ &= \frac{1}{k}\ \text{curl}\left[\mathbf{e}_z\left\{\sum_{l=m}^{\infty} i^{m-l}\,(2l+1)\frac{(l-m)!}{(l+m)!}\;P_l^m\left(\pm\frac{ib}{k_\beta}\right)\Psi_{ml}\right\}\right]\\ &= \frac{1}{k}\sum_{l=m}^{\infty} i^{m-l}\,(2l+1)\,\frac{(l-m)!}{(l+m)!}\;P_l^m\left(\pm\frac{ib}{k_\beta}\right)\text{curl}\,(\mathbf{e}_z\Psi_{ml}).\end{aligned}$$

$\ldots(12)$

Furthermore, we know that (Ben-Menahem and Singh 1981, p. 1043)

$$\begin{aligned}\frac{1}{k_\beta}\ \text{curl}\,(\mathbf{e}_z\Psi_{ml}) &= \frac{im}{l(l+1)}\,\mathbf{N}_{ml} + \frac{l-m+1}{(l+1)(2l+1)}\,\mathbf{M}_{m,l+1}\\ &\quad + \frac{l+m}{l(2l+1)}\,\mathbf{M}_{m,l-1}\,,\ l \geqslant 1\,,\\ \frac{1}{k_\beta}\ \text{curl}\,(\mathbf{e}_z\Psi_{00}) &= \mathbf{M}_{01}\,. \qquad \ldots(13)\end{aligned}$$

Equations (12) and (13) give

$$\mathbf{M}_m^{\pm} = \sum_{l=m}^{\infty} i^{m-l}\,(2l+1)\,\frac{(l-m)!}{(l+m)!}\;P_l^m\left(\pm\frac{ib}{k_\beta}\right)\frac{k_\beta}{k}\left[\frac{im}{l(l+1)}\,\mathbf{N}_{ml}\right.$$

*(equation continued on p. 178)*

$$+ \frac{l-m+1}{(l+1)(2l+1)} \mathbf{M}_{m,l+1} + \frac{l+m}{l(2l+1)} \mathbf{M}_{m,l-1} \Big] . \quad \ldots(14)$$

For $m = 0$, the first term in the summation on the right-hand side of (14) is $\frac{k_\beta}{k} \mathbf{M}_{01}$.

Equation (14) gives the expansion of the cylindrical vector $\mathbf{M}_m^{\pm}$ in terms of the spherical vectors $\mathbf{M}_{ml}$ and $\mathbf{N}_{ml}$. To find the corresponding expansion for $\mathbf{N}_m^{\pm}$, we note from (3), (4), (7) and (8) that

$$\left.\begin{aligned} \text{curl } \mathbf{M}_m^{\pm} &= k_\beta \mathbf{N}_m^{\pm} \\ \text{curl } \mathbf{M}_{ml} &= k_\beta \mathbf{N}_{ml} \\ \text{curl } \mathbf{N}_{ml} &= k_\beta \mathbf{M}_{ml}. \end{aligned}\right\} \quad \ldots(15)$$

Therefore, taking the curl of (14) and using (15), we find

$$\mathbf{N}_m^{\pm} = \sum_{l=m}^{\infty} i^{m-l} (2l+1) \frac{(l-m)!}{(l+m)!} P_l^m \left( \pm \frac{ib}{k_\beta} \right) \frac{k_\beta}{k} \Big[ \frac{im}{l(l+1)} \mathbf{M}_{ml}$$

$$+ \frac{l-m+1}{(l+1)(2l+1)} \mathbf{N}_{m,l+1} + \frac{l+m}{l(2l+1)} \mathbf{N}_{m,l-1} \Big] . \quad \ldots(16)$$

For $m = 0$, the first term in the summation on the right-hand side of (16) is $\frac{k_\beta}{k} \mathbf{N}_{01}$.

Equations (11), (14) and (16) constitute our final results. These express the cylindrical vector solutions of the dynamical Navier equation (1) in terms of the spherical vector solutions. It may be noted that while the longitudinal solution $\mathbf{L}_m^{\pm}$ can be expressed in terms of $\mathbf{L}_{ml}$ alone, for the expansion of a transverse solution $\left(\mathbf{M}_m^{\pm} \text{ or } \mathbf{N}_m^{\pm}\right)$ we need both $\mathbf{M}_{ml}$ and $\mathbf{N}_{ml}$. The expansions obtained are useful when computations can be easily carried out in spherical coordinates but computations in cylindrical coordinates are tedious.

## REFERENCE

Ben-Menahem, A., and Singh, S. J. (1981). Seismic Waves and Sources. Springer-Verlag, New York.