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ON SOME APPROXIMATION PROBLEMS IN METRIC LINEAR SPACES

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In this paper we consider the problem of simultaneous characterization of a set of elements of best approximation and characterization of the elements of ϵ-approximation in metric linear spaces.

1. INTRODUCTION

The convexity of spheres in normed linear spaces and the existence of non trivial topological dual spaces of normed linear spaces have helped a lot in developing a fairly large theory of approximation in normed linear spaces (see e.g. Singer 1970). On the other hand, it has not been possible to do so in metric linear spaces inspite of the best efforts of many mathematicians as in these spaces the spheres are not generally convex, there are spaces whose dual spaces consist of zero element alone e.g. the space l^p, $0 < p < 1$ has no base of convex neighbourhoods and so not locally convex and 0 is the only continuous linear functional on the space L^p, $0 < p < 1$ (it has no convex open sets other than ϕ and L^p) etc. Because of the lack of the duality theory in metric linear spaces it is not possible sometime to apply Hahn-Banach theorem which is an important tool in proving many results on approximation in normed linear spaces. The authors, in a series of papers, have also made an attempt in developing a theory of best approximation in metric linear spaces and the present paper is a step in the same direction.

In this paper we consider the problem of simultaneous characterization of a set of elements of best approximation and characterization of the elements of ϵ-approximation in metric linear spaces.

2. SIMULTANEOUS CHARACTERIZATION OF SET OF ELEMENTS OF BEST APPROXIMATION

In metric linear spaces the problem of characterization of elements of best approximation was considered by Pantelidis (1969). A natural generalization of this problem is simultaneous characterization of a set of elements of best approximation :

Given a metric linear space (E, d), G a linear subspace of E, $x \in E \mid \bar{G}$ and a subset M of G, what are the necessary and sufficient conditions in order that every element $g_0 \in M$ be an element of best approximation to x by the elements of G? We will answer this in Theorem 2.2. As a consequence of this we get a characterization of semi-Chebyshev subspaces (Corollary 2.1).

Pantelidis (1969) gave the following characterization of elements of best approximation in metric linear spaces. The corresponding characterization in normed linear spaces was given by Singer (1970) as a simple application of Hahn-Banach theorem (cf. Theorem 1. 1, Chapter I).

Theorem 2.1—If G is a closed linear subspace of real metric linear space (E, d) and $x_0 \in E \mid G$. Then $g_0 \in \alpha_G(x_0)$

$$\alpha_G(x_0) = \{ g \in G : d(x_0, g) = d(x_0, G) \}$$

if and only if there exists an $f \in E^v$,

$$E^v = \left\{ f : E \to \mathbf{R} : \sup_{x \in E/\{0\}} \frac{|f(x)|}{d(x, 0)} < \infty, \right.$$

$$\left. f(0) = 0, f \text{ subadditive} \right\}$$

with the following properties :

(i) $|f(x) - f(y)| \leqslant d(x, y)$, $x, y \in E$

(ii) $f(x + g) = f(x)$, $x \in E$, $g \in G$ or $f_G = 0$

(iii) $f(x_0 - g_0) = f(x_0) = d(x_0, g_0)$.

The following theorem gives simultaneous characterization of a set of elements of best approximation in metric linear spaces. In normed linear spaces such a characterization was given by Singer (1970, Corollary 1.3, Chapter I).

Theorem 2.2 — Let (E, d) be a real metric linear space, G a closed linear subspace of E, $x \in E/G$ and $M \subset G$. Then $M \subset \alpha_G(x)$ if and only if there exists $f \in E^v$ satisfying

(i) $|f(x) - f(y)| \leqslant d(x, y)$

(ii) $f_G = 0$

(iii) $f(x - g) = d(x, g)$ for all $g \in M$.

PROOF : Suppose that $M \subset \alpha_G(x)$ and let $g_0 \in M$. Then $g_0 \in \alpha_G(x)$ and so by Theorem 2.1, there exists $f \in E^v$ such that

$$|f(x) - f(y)| \leqslant d(x, y) \quad \text{for all } x, y \in E, f_G = 0$$

and

$$f(x - g_0) = d(x, g_0), \quad g_0 \in M.$$

Now let $g \in M$. Then $g \in \alpha_G(x)$. Consider

$$f(x-g) = f(x-g_0)$$
$$= d(x, g_0)$$
$$= d(x, g), \text{ as } d(x, g_0) = d(x, g) = d(x, G).$$

Conversely, let us suppose that there exists $f \in E^v$ satisfying (i), (ii) and (iii) and let $g_0 \in M$. Then by Theorem 2.1, $g_0 \in \alpha_G(x)$. Hence $M \subset \alpha_G(x)$.

As as a consequence of this we get the following characterization of semi-Chebyshev subspaces of real metric linear space (A linear subspace of G of a metric linear space (E, d) is called semi-Chebyshev if for every $x \in E$, the set $\alpha_G(x)$ contains at most one elements).

Corollary 2.1 — Let (E,d) be a real metric linear space and G a linear subspace of E. Then the following statements are equivalent

(1) G is semi-Chebyshev;

(2) there exists no $f \in E^v$ and $x_1, x_2 \in E$ with $x_1 - x_2 \in G/\{0\}$ such that

(a) $|f(x)| \leqslant d(x, 0), \quad x \in E$

(b) $f_G = 0$

(c) $f(x_1) = d(x_1, 0)\ f(x_2), = d(x_2, 0)$;

(3) there exists no $f \in E^v$, $x \in E$ and $g_0 \in G / \{0\}$ with the properties *(a)*, *(b)* and *(d)* $f(x) = d(x, 0) = d(x, g_0)$.

The proof of this is exactly similar to that given in Singer (1970, Theorem 3.2, Chapter I) for normed linear spaces.

An independent proof of this was given by Pantelidis (1969, Theorem 4.2).

3. Characterization of Elements of ϵ Approximation

If (E, d) is a metric linear space, G a linear subspace of E, $x \in E$ and $\epsilon > 0$, an element $g_0 \in G$ is said to be an element of ϵ-approximation of x (by means of the elements of G) if we have $d(x, g_0) \leqslant d(x, G) + \epsilon$. We shall denote by $\alpha_G(x, \epsilon)$, the set of all elements of ϵ-approximation of x. One of the advantages of considering the sets $\alpha_G(x, \epsilon)$ with $\epsilon > 0$, instead of the sets $\alpha_G(x)$, is that the sets $\alpha_G(x, \epsilon)$ are always non-void for $\epsilon > 0$.

In normed linear spaces the elements of ϵ-approximation and the sets $\alpha_G(x, \epsilon)$ have been introduced by Buck (1965) who used the term 'Elements of good approximation' instead of 'elements of ϵ-approximation'. Similar to characterization of elements of ϵ-approximation in normed linear spaces (cf. Singer 1970, Theorem 6.12, Chapter I), we have the following theorem which gives a characterization of elements of ϵ-approximation in metric linear spaces.

Theorem 3.1 — Let G be a linear subspace of a real metric linear space (E, d) and $x_0 \in E/G$. Then $g_0 \in \alpha_G(x_0, \epsilon)$ if and only if there exists an $f \in E^v$ such that

(i) $\quad |f(x) - f(y)| \leqslant d(x, y), \quad x, y \in E$

(ii) $f(g) = 0$ for all $g \in G$

(iii) $f(x_0 - g_0) \geqslant d(x_0, g_0) - \epsilon$.

PROOF : Let us suppose that $g_0 \in \alpha_G(x_0, \epsilon)$. Define $f(x) = d(x, G)$. Then clearly $f(g) = d(g, G) = 0$ for all $g \in G$ and therefore (ii) is satisfied.

Since $g_0 \in \alpha_G(x_0, \epsilon)$, $d(x_0, g_0) \leqslant d(x_0, G) + \epsilon$ i. e. $d(x_0, g_0) - \epsilon \leqslant d(x_0, G)$ $f(x_0) = f(x_0 - g_0)$ and hence (iii) is proved. Consider $f(x + y) = d(x + y, G)$ $= \inf_{g \in G} d(x + y, g') = \inf_{g, g' \in G} d(x + y, g + g') = \inf_{g, g' \in G} d(x - g, -y + g') \leqslant \inf_{g, g' \in G}$ $\{ d(x - g, 0) + d(0, -y + g') \} = \inf_{g, g' \in G} \{ d(x, g) + d(y, g') \} \leqslant \inf_{g \in G} d(x, g)$ $+ \inf_{g' \in G} d(y, g') = d(x, G) + d(y, G) = f(x) + f(y)$.

Therefore $f(x + y) \leqslant f(x) + f(y)$, consequently $f \in E^v$. Now $f(x) = d(x, G)$ $= \inf_{g \in G} d(x, g) \leqslant \inf_{g \in G} \{ d(x, y) + d(y, g) \} \leqslant \inf_{g \in G} d(x, y) + \inf_{g \in G} d(y, g)$ $= d(x, y) + d(y, G) = d(x, y) + f(y)$.

i. e. $f(x) \leqslant d(x, y) + f(y)$

i. e. $f(x) - f(y) \leqslant d(x, y)$. Interchanging x and y, we get

$$f(y) - f(x) \leqslant d(y, x) = d(x, y).$$

Hence $|f(x) - f(y)| \leqslant d(x, y)$ i. e. (1) is proved.

Convesely, let us suppose that there exists an $f \in E^v$ satisfying the conditions (i), (ii) and (iii). Then

$$\begin{aligned} d(x_0\, g_0) &\leqslant f(x_0 - g_0) + \epsilon \\ &= f(x_0 - g) + \epsilon \text{ for all } g \in G \\ &\leqslant |f(x_0 - g)| + \epsilon \text{ for all } g \in G \\ &\leqslant |f(x_0) - f(g)| + \epsilon \text{ for all } g \in G \\ &\leqslant d(x_0, g) + \epsilon \text{ for all } g \in G. \end{aligned}$$

This implies that $d(x_0, g_0) \leqslant d(x_0, G) + \epsilon$. Hence $g_0 \in \alpha_G(x_0, \epsilon)$.

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