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DUALITY THEORY FOR GENERALIZED FRACTIONAL PROGRAMMES

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This paper presents a duality theory for generalized fractional programmes and the duals are a form of inexact linear programmes which can be solved by the simplex method.

INTRODUCTION

In the modelling of a given problem by means of a fractional programme

$$\text{Min} \left\{ \frac{cx + \alpha}{dx + \beta} \;\middle|\; Ax = b,\ x \geqslant 0 \right\}$$

the coefficients b_i, a_{ij}, c_j and d_j are assumed to be known exactly. In practice, however, this is seldom the case. Sensitivity analysis, chance constrained and stochastic programming approaches do not require exact knowledge of the coefficients. However, in many cases, it is possible to determine that the activity vectors belong to a given set.

In this paper we consider the generalized linear fractional programme (GLFP)

$$\text{Min} \quad \frac{cx + \alpha}{dx + \beta}$$

$$\text{Subject to } a_1x_1 + a_2x_2 + \ldots + a_nx_n = b \qquad \ldots(1)$$

$$x_j \geqslant 0$$

where each column vector $a_j \in K_j$, $j = 1, 2, \ldots, n$, K_j is a convex polyhedron in R^m and we assume that

(i) $dx + \beta > 0$ over the set S of feasible solutions of (1)

(ii) S is regular (i.e. non empty and bounded).

This problem can be examined from two different point of views. We can seek the optimal solution which is feasible for all possible $a_j \in K_j$. This approach is called inexact fractional programming and has been considered by Kaur (1981). We can also view the optimal solution as the best possible value of the problem i.e. we seek the optimal solution where the solution is feasible for some $a_j \in K_j$. In this

case the activity vectors a_j along with the vector x are the decision variables. This paper presents duality and computational results for the latter strategy. Such problems have also been treated by Kaul and Bhatia (1974) where the optimal solution is obtained by applying decomposition method. Duality and computational results for the generalized linear programming problem have recently been developed by Thuente (1980).

Dual Problems

Let K_j, $j = 1, 2, \ldots, n$, be bounded polyhedral convex sets in R^m and let a_j denote the columns of a matrix A with a_j from K_j.
We can write GLFP in the form

$$\text{Min } \frac{cx + \alpha}{dx + \beta} \cdot \qquad \ldots(2)$$

$$\text{Subject to } Ax = b \text{ where } Ax = b \text{ for some } a_j \in K_j,$$

$$x \geqslant 0.$$

This will be called the primal problem. An optimal soultion to (2) is a pair (x^*, A^*) such that

$$A^*x^* = b,\ x^* \geqslant 0 \text{ and } \frac{cx^* + \alpha}{dx^* + \beta} \leqslant \frac{cx + \alpha}{dx + \beta}$$

for all other pairs (x, A) such that $Ax = b$ and $x \geqslant 0$.
We assert that dual problem of (2) has the form of an inexact linear programme

$$\text{Max } ew$$

$$\text{Subject to } w\hat{A} \leqslant \hat{c} \text{ for all } a_j \in K_j \qquad \ldots(3)$$

$$\text{where } \hat{A} = \begin{pmatrix} A & -b \\ d & \beta \end{pmatrix},\ \hat{c} = (c, \alpha)$$

$$e = (0, 1),\ 0 \in R^m \text{ and } w \in R^{m+1}.$$

We shall show that (2) and (3) are primal dual problems in the usual sense of linear programming.

Theorem (Duality Theorem)—If either the primal (2) or the dual (3) has a finite optimal solution then the other problem has a finite optimal solution and their optimal values are same.

Proof : Since each K_j is a bounded polyhedral convex set, any point in K_j can be written as convex combination of its extreme points $\left(a_j^1, a_j^2, \ldots, a_j^{r_j} \right)$.
For any column vector $a_j \in K_j$,

$$a_j = \sum_{i=1}^{r_j} \alpha_j^i \text{ where } \sum_{i=1}^{r_j} \alpha_i = 1.$$

We can rewrite the programme (2) as

$$\text{Min } \frac{c_1x_1 + c_2x_2 + \ldots + c_nx_n + \alpha}{d_1x_1 + d_2x_2 + \ldots + d_nx_n + \beta}$$

Subject to

$$\sum_{i=1}^{r_1} \alpha_i a_1^i x_1 + \sum_{i=1}^{r_2} \alpha_i a_2^i x_2 + \ldots + \sum_{i=1}^{r_n} \alpha_i a_n^i x_n = b \qquad \ldots(4)$$

$$x_j \geqslant 0, \sum_{i=1}^{r_j} \alpha_i = 1 \text{ for each } j = 1, 2, \ldots, n.$$

Let $x_j^i = \alpha_i x_j$ then $x_j = \sum_{i=1}^{r_j} x_j^i$.

Thus (4) is equivalent to

$$\text{Min } \frac{c_1 \sum_{i=1}^{r_1} x_1^i + c_2 \sum_{i=1}^{r_2} x_2^i + \ldots + c_n \sum_{i=1}^{r_n} x_n^i + \alpha}{d_1 \sum_{i=1}^{r_1} x_1^i + d_2 \sum_{i=1}^{r_2} x_2^i + \ldots + d_n \sum_{i=1}^{r_n} x_n^i + \beta} \qquad \ldots(5)$$

Subject to

$$\sum_{i=1}^{r_1} a_1^i x_1^i + \sum_{i=1}^{r_2} a_2^i x_2^i + \ldots + \sum_{i=1}^{r_n} a_n^i x_n^i = b$$

$$x_j^i \geqslant 0,\ i = 1, 2, \ldots, r_j,\ j = 1, 2, \ldots, n.$$

Dual to (5), according to Schiable (1976), is the following linear programming problem

Max z

Subject to $ua_j^i + d_j z \leqslant c_j$ for each $i = 1, 2, \ldots, r_j,\ j = 1, 2, \ldots, n,$

$$-ub + \beta z \leqslant \alpha. \qquad \ldots(6)$$

$$u \in R^m,\ z \in R.$$

Since $\left(a_j^1, a_j^2, \ldots, a_j^{r_j} \right)$ are the extreme points of K_j and they generate it, we can write the first constraint of (6) in the form

$$ua_j \leqslant c_j - d_j z \text{ for each } j = 1, 2, \ldots, n, \forall\ a_j \in K_j$$

because $\sum_{i=1}^{r_j} \alpha_i = 1$ for each $j = 1, 2, \ldots, n$.

Thus dual (6) of (5) can be written as

$$\begin{aligned} &\text{Max } z \\ &\text{Subject to } uA + dz \leqslant c \\ &\qquad -ub + \beta z \leqslant \alpha \qquad \ldots(7) \\ &\qquad u \in R^m,\ z \in R. \end{aligned}$$

Taking

$$w = (u, z),\ \hat{c} = (c, \alpha),\ \hat{A} = \begin{pmatrix} A & -b \\ d & \beta \end{pmatrix}$$

and $e = (0, 1)$ where $0 \in R^m$, we can write the dual (7) in the form

$$\begin{aligned} &\text{Max } ew \\ &\text{Subject to } w\hat{A} \leqslant \hat{c} \text{ for all } a_j \in K_j \end{aligned}$$

which is same as programme (3). Thus if either (2) or (3) has a finite optimal solution then the other also has a finite optimal solution and they have the same optimal values on account of the fact that (5) is equivalent to (2) and the dual (6) of (5) can be put in the form (3). This concludes the proof.

When the activity sets K_j are parallelopipeds, the dual can be simplified further.

Corollary—Let $K_j = \{ a_j \mid \underline{a_j} \leqslant a_j \leqslant \overline{a_j} \}$, let $\underline{A}$ and $\overline{A}$ be matrices with columns $\underline{a_j}$ and $\overline{a_j}$ respectively and let

$$\underline{\hat{A}} = \begin{pmatrix} \underline{A} & -b \\ d & \beta \end{pmatrix},\quad \overline{\hat{A}} = \begin{pmatrix} \overline{A} & -b \\ d & \beta \end{pmatrix}$$

then the dual of (2) is given by

$$\begin{aligned} &\text{Max } w'e - w''e \\ &\text{Subject to} \\ &w'\,\overline{\hat{A}} - w''\,\underline{\hat{A}} \leqslant \hat{c} \qquad \ldots(8) \\ &w',\ w'' \geqslant 0. \end{aligned}$$

The above corollary can be proved on the same lines as corollary given in section 1 of Thuente's paper (1980) with A replaced by $\hat{A}$.

COMPUTATIONAL METHOD

The solution to GLFP (2) is a pair (x^*, A^*). It was shown that (2) is equivalent to (5) and if x_j^i are the primal variables in (5) then $x_j = \sum_{i=1}^{r_j} x_j^i$. Thus in order to find x^*, solve (5). If $\left(x_j^{*i} \right)$ is the solution of (5) then $x_j^* = \sum_{i=1}^{r_j} x_j^{*i}$. Further, a_j^* can be defined as in Thuente (1980).

If the activity sets K_j are parallelopipeds then the programme (8) gives an effective method of computing the optimal value of programme (2) and the dual variables to (8) give the optimal solution x^* of (2). If we use programme (5) or (6) to solve the GLFP for these K_j, we shall have to consider 2^m extreme points for each K_j with corresponding number of variables or constraints. However, in (8), we need to consider only the largest and smallest extreme points of each K_j. Thus to solve this special case, we use programme (8) or its dual. The programme (8) can be written in the form

$$\begin{aligned} &\text{Max } z' - z'' \\ &\text{Subject to } \bar{A}u' - \underline{A}u'' + d(z' - z'') \leqslant c \\ &\qquad b(u'' - u') + \beta(z' - z'') \leqslant \alpha \\ &\qquad u', u'', z', z'', \geqslant 0 . \end{aligned} \qquad \ldots(9)$$

Now dual to (9) is

$$\begin{aligned} &\text{Min } cV + \alpha\eta \\ &\text{Subject to } \quad \bar{A}V - b\eta \geqslant 0 \\ &\qquad\qquad -\underline{A}V + b\eta \geqslant 0 \\ &\qquad\qquad dV + b\eta = 1 \\ &\qquad\qquad V, \eta \geqslant 0. \end{aligned} \qquad \ldots(10)$$

which by Charnes and Cooper (1962) is equivalent to the following programming problem

$$\begin{aligned} &\text{Min } \frac{cx + \alpha}{dx + \beta} \\ &\text{Subject to } \quad \bar{A}x \geqslant b \\ &\qquad\qquad \underline{A}x \leqslant b \\ &\qquad\qquad x \geqslant 0. \end{aligned} \qquad \ldots(11)$$

Thus in this case we solve programme (9) and if $(u'^*, u''^*, z'^*, z''^*)$ is the optimal solution of (9) and (V^*, η^*) is the dual optimal solution of (9) then $x^* = \frac{V^*}{\eta^*}$ is an

optimal solution of (2). The above conclusion follows from the fact that (10) and (11) are equivalent, (10) is a dual of (8) which in its turn is the dual of (2).

The optimal A^* can be constructed by using the optimal solution of (9). Defining $\underline{b_i}$, $\overline{b_i}$ and choosing λ and $(1 - \lambda)$ as in (Thuente 1980) the optimal A^* can be given by

$$a_{ij}^* = \begin{cases} \overline{a_{ij}} & \text{if } u_i'^* > 0 \\ \underline{a_{ij}} & \text{if } u_i''^* > 0 \\ \lambda \underline{a_{ij}} + (1 - \lambda)\,\overline{a_{ij}} & \text{if } u_i'^* = u_i''^* = 0 \text{ and } \underline{b_i} \neq \overline{b_i} \\ \overline{a_{ij}} & \text{if } \underline{b_i} = \overline{b_i}\,. \end{cases} \qquad \ldots(12)$$

The following examples illustrate the above method in both the cases.

Example 1

$$\text{Min} \quad \frac{3x_1 + 2x_2 + 2x_3}{x_2 + x_3 + 2}$$

$$\text{Subject to} \quad a_1x_1 + a_2x_2 + a_3x_3 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \ldots(13)$$

$$x_1,\ x_2, x_3 \geqslant 0$$

$$\text{for some } a_j \in K_j,\ J = 1, 2, 3$$

where K_j are given by

$$K_1 : u_1 + u_2 \leqslant 1$$
$$u_1 + 3u_2 \leqslant 3$$
$$u_1, u_2 \geqslant 0$$
$$K_2 : v_1 + v_2 \geqslant 2$$
$$v_1 + 2v_2 \leqslant 4$$
$$v_1, v_2 \geqslant 0$$
$$K_3 : 2w_1 + w_2 \leqslant 3$$
$$w_1 + 2w_2 \leqslant 4$$
$$4w_1 + 3w_2 \geqslant 6$$
$$w_1, w_2 \geqslant 0.$$

The extreme points of K_j are given by

$$K_1 : \{(1, 0),\ (3, 0),\ (0, 1)\}$$

$$K_2 : \{(2, 0),\ (4, 0),\ (0, 2)\}$$

$$K_3 : \left\{\left(\frac{3}{2}, 0\right), \left(0, 2\right), \left(\frac{2}{3}, \frac{5}{3}\right)\right\}.$$

On solving programme (5), we get the optimal solution

$$x_2^{*2} = \frac{1}{4}, \quad x_2^{*3} = 1 \quad \text{with all other} \quad x_j^{*l} = 0.$$

Then the solution of (13) is given by

$$x_1^* = 0, \; x_2^* = \frac{1}{4}, \; x_3^* = 1$$

and

$$a_1^* = \left[\frac{1}{2} \quad + \frac{1}{2}\right], \quad a_2^{\epsilon} = [0, 2], \; a_3^* = [4, 0].$$

Example 2

$$\text{Min} \quad \frac{x_1 + 4x_2 + 3x_3}{x_1 + 2x_2 + 3x_3 + 4}$$

$$\text{Subject to } a_1x_1 + a_2x_2 + a_3x_3 = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} \qquad \ldots(14)$$

$$x_1, x_2, x_3 \geqslant 0$$

for some $a_j \in K_j, j = 1, 2, 3$

where K_j are given by

$$K_1 : [1, -1, 2] \leqslant a_1 \leqslant [1, 0, 3]$$

$$K_2 : [2, -3, 1] \leqslant a_2 \leqslant [3, 1, 2]$$

$$K_3 : [0, 1, -2] \leqslant a_3 \leqslant [1, 2, -1].$$

The columns of $\bar{A}$ ($\underline{A}$) are the upper (lower) bounds on the a_j. Using (9) to solve (14) we get the optimal solution $u_1'^* = \frac{16}{21}$, $u_3''^* = \frac{4}{21}$, $z'^* = \frac{13}{21}$ and all other variables are zero. Its dual solution is given by

$$v_1^* = \frac{11}{42}, \; v_2^* = 0, \; v_3^* = \frac{5}{42} \text{ and } \eta^* = \frac{2}{21}.$$

Thus optimal solution of the given problem (14) is given by

$$x_1^* = \frac{11}{4}, \quad x_2^* = 0, \; x_3^* = \frac{5}{4}.$$

Since $u_1'^*$ and $u_3''^*$ are positive, the first and third rows of A^* are from $\bar{A}$ and $\underline{A}$ respectively. Since $u_2'^* = 0 = u_2''^*$, therefore 2nd row is to be calculated by using $\underline{b_i}$ and $\overline{b_i}$.

Thus, by (12),

$$A^* = \begin{pmatrix} 1 & 3 & 1 \\ -\frac{1}{8} & \frac{1}{2} & \frac{15}{8} \\ 2 & 1 & -2 \end{pmatrix}$$

Also $A^*x^* = b$ and the optimal value of the problem (14) is $\frac{13}{21}$.

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