*Indian J. pure appl. Math.*, **14**(3) : 398–411, March 1983

# FINITE DIFFERENCE METHODS FOR A CERTAIN TWO POINT BOUNDARY VALUE PROBLEM

RIAZ A USMANI

*Department of Applied Mathematics, University of Manitoba, Winnipeg, Manitoba, Canada R3T 2N2*

(*Received* 30 *January* 1982)

In this paper, finite difference methods of orders 2, 4 and 6, for the numerical solution of a two-point boundary value problem associated with a fourth-order linear ordinary differential equation, are developed and analysed. A sufficient condition guaranteeing a unique solution of the corresponding boundary value problem is also given. Numerical results are tabulated for two typical numerical examples and compared with the shooting technique employing the classical fourth-order Runge-Kutta method.

## 1. INTRODUCTION

We consider in this report some finite difference methods for the numerical approximation of the real solution of the two point boundary value problem

$$\begin{aligned} &[L + f(x)]\, y(x) = g(x), \quad L \equiv d^4/dx^4, \; a < x < b \\ &y(a) - A_1 = y'(a) - A_2 = 0 \\ &y''(b) - B_1 = y'''(b) - B_2 = 0 \end{aligned} \qquad \ldots(1.1)$$

where the functions $f(x)$ and $g(x)$ are continuous on $[a, b]$ and $A_i$, $B_i$ $(i = 1, 2)$ are finite real arbitrary constants. The analytical solution of (1.1) for arbitrary choices of $f(x)$ and $g(x)$ cannot be determined. We overcome this difficulty by resorting to discrete variable methods for obtaining an approximate solution of the differential system (1.1). The purpose of this note is two-fold. We first examine the conditions under which the systems (1.1) has a unique solution. We then pass on to develop and analyse three different finite difference methods of orders 2, 4 and 6 respectively, to obtain an approximate solution of (1.1). For the sake of comparison we also solve (1.1) by a modified shooting technique which will be described in section 5. In the end numerical evidence is included to demonstrate the superiority and practical usefulness of our finite difference schemes. We remark that the results of this note are also applicable to a nonlinear differential system of the form

This work was supported in part by the National Scientific & Engineering Research Council of Canada Grant No. A 5272.

$$Ly = \phi(x, y),\ y(a) - A_1 = y'(a) - A_2 = y''(b) - A_3 = y'''(b) - A_4 = 0. \quad \ldots(1.2)$$

## 2. Uniqueness of the Solution of (1.1)

We first give a simple sufficient condition for a unique solution of the differential system (1.1).

*Theorem* 2.1—The boundary value problem (1.1) has a unique solution provided

$$f(x) \geqslant 0,\ f(x) \not\equiv 0,\ x \in [a, b]. \quad \ldots(2.1)$$

Proof: We show that the system (1.1) has one and only one solution. Assume that $u(x)$ and $v(x)$ are any two distinct solutions of the system (1.1). Then it follows that $w(x) = u(x) - v(x)$ will satisfy the differential system

$$[L + f(x)]\, w(x) = 0,\ w(a) = w'(a) = w''(b) = w'''(b) = 0. \quad \ldots(2.2)$$

Multiply the differential equation $[L + f(x)]\, w(x) = 0$ in (2.2) by $w(x)$, integrate twice from $a$ to $b$, and use the boundary conditions in (2.2) to obtain

$$\int_a^b [w''(x)]^2\, dx + \int_a^b f(x)\, [w(x)]^2\, dx = 0. \quad \ldots(2.3)$$

Since both the definite integrals on the left hand side of the equality sign in (2.3) are nonnegative, it follows that

$$\text{(i)} \quad \int_a^b [w''(x)]^2\, dx = 0,$$

$$\text{(ii)} \quad \int_a^b f(x)\, [w(x)]^2\, dx = 0. \quad \ldots(2.4)$$

The first integral (2.4i) implies that $w''(x) = 0$ almost everywhere (a.e.) on $[a, b]$, or $w(x)$ is at most a linear function of the form $w(x) = \alpha + \beta x$. But the second integral (2.4ii) implies $w(x) \equiv 0$ a. e. unless $f(x) \equiv 0$ a. e. which is excluded in view of hypothesis (2.1).

Thus, the homogeneous differential system (2.2) has only the trivial solution $w(x) \equiv 0$, or $u(x) \equiv v(x)$ and the system (1.1) has at most one solution.

## 3. Finite Difference Methods

We first introduce a finite set of grid points $\{x_n\} \subset [a, b]$

$$x_n = a + nh,\ n = 0, 1, 2, \ldots, N, \quad \ldots(3.1)$$

where $h = (b - a)/N$. Here, naturally, $N$ is a suitable positive integer.

Our basic finite difference formula can be derived from

$$h^4 y_n^{(4)} = \left( \delta^4 - \frac{\delta^6}{6} + \frac{7\delta^8}{240} - \frac{41\delta^{10}}{7560} + \ldots \right) y_n,\ n = 2, 3, \ldots, N - 2. \quad \ldots(3.2)$$

The formula (3.2) is given by Fox (1957, p.19). From (3.2) we obtain our scheme

$$h^4 y_n^{(4)} = \delta^4 y_n + \frac{1}{6} h^6 y^{(6)}(\xi_n),\ n = 2(1)\ N-2,\ x_{n-2} < \xi_n < x_{n+2}. \qquad \ldots(3.3)$$

Here we write $y_n$ for $y(x_n)$ and we assume that $y(x)$ is the exact solution of the system (1.1). We further note that from (3.3), on neglecting truncation error, and setting $y_n \simeq z_n$, we obtain

$$\delta^4 z_n = h^4 z_n^{(4)},\ n = 2(1)\ N-2 \qquad \ldots(3.4)$$

where $z_n^{(4)} = -f_n z_n + g_n$, and $f_n \equiv f(x_n)$, $g_n \equiv g(x_n)$.

The system (3.4) give us $(N-3)$ equations in the $N$ unknowns $z_n$, $n = 1(1)N$. Therefore, we develop three more formulas, (one at the boundary $x = a$ and two at the boundary $x = b$), namely,

$$\text{(i)} \quad -\frac{11}{2} y_0 + 9y_1 - \frac{9}{2} y_2 + y_3 = 3hy_0' + \frac{h^4}{20}\left[-3y_0^{(4)} + 18y_1^{(4)}\right] + \frac{7}{40} h^6 y^{(6)}(\xi_1);$$

$$\text{(ii)} \quad y_{N-3} - 4y_{N-2} + 5y_{N-1} - 2y_N = -h^2 y_N'' + h^4\left[y_{N-1}^{(4)} - \frac{1}{12} y_N^{(4)}\right] + \frac{59}{360} h^6 y^{(6)}(\xi_{N-1});$$

$$\text{(iii)} \quad y_{N-2} - 2y_{N-1} + y_N = h^2 y_N'' - h^3 y_N''' + h^4\left[\frac{1}{4} y_{N-1}^{(4)} + \frac{1}{3} y_N^{(4)}\right] + \frac{2}{45} h^6 y^{(6)}(\xi_N); \qquad \ldots(3.5)$$

where $x_0 < \xi_1 < x_3$, $x_{N-3} < \xi_{N-1} < x_N$, and $x_{N-2} < \xi_N < x_N$.

From (3.5), on neglecting the respective truncation errors, and setting $y_n \simeq z_n$, we obtain 3 more equations in the $z_n$ unknowns, namely,

$$\text{(i)} \quad -\frac{11}{2} z_0 + 9z_1 - \frac{9}{2} z_2 + z_3 = 3hz_0' + \frac{h^4}{20}\left[-3z_0^{(4)} + 18z_1^{(4)}\right];$$

$$\text{(ii)} \quad z_{N-3} - 4z_{N-2} + 5z_{N-1} - 2z_N = -h^2 B_1 + h^4\left[z_{n-1}^{(4)} - \frac{1}{12} z_N^{(4)}\right];$$

$$\text{(iii)} \quad z_{N-2} - 2z_{N-1} + z_N = h^2 B_1 - h^3 B_2 + h^4\left[\frac{1}{4} z_{N-1}^{(4)} + \frac{1}{3} z_N^{(4)}\right]. \qquad \ldots(3.6)$$

Thus, eqns. (3.6 i), (3.4), (3.6 ii), and (3.6 iii) in the $N$ unknowns $z_N$ are solved to obtain an approximate solution of our original boundary value problem. We shall prove later that the procedure just described for obtaining an approximate solution of (1.1) is a second order method. We also note that the matrix associated with the system of linear equations in the $N$ unknown $z_n$, $n = 1(1)N$, is a five-band matrix.

Our fourth order finite difference method is based on the formulas

$$\text{(i)} \quad -\frac{11}{2}y_0 + 9y_1 - \frac{9}{2}y_2 + y_3 = 3hy_0' + \frac{h^4}{280}\left[8y_0^{(4)} + 151y_1^{(4)} + 52y_2^{(4)} - y_3^{(4)}\right] + \frac{h^8}{6720}y_0^{(8)} + O(h^9);$$

$$\text{(ii)} \quad \delta^4 y_n = \frac{h^4}{6}\left[y_{n-1}^{(4)} + 4y_n^{(4)} + y_{n+1}^{(4)}\right] - \frac{h^8}{720}y_n^{(8)} + O(h^9),\ n = 2(1)N-2;$$

$$\text{(iii)} \quad y_{N-3} - 4y_{N-2} + 5y_{N-1} - 2y_N = -h^2y_N'' + \frac{h^4}{360}\left[y_{N-3}^{(4)} + 56y_{N-2}^{(4)} + 245y_{N-1}^{(4)} + 28y_N^{(4)}\right] - \frac{241}{60480}y^{(8)}(\xi_{N-1}),$$

$$\xi_{N-1} \in (x_{N-3}, x_N);$$

$$\text{(iv)} \quad y_{N-2} - 2y_{N-1} + y_N = h^2y_N'' - h^3y_N''' + \frac{h^4}{180}\left[4y_{N-2}^{(4)} - 19y_{N-2}^{(4)} + 71y_{N-1}^{(4)} + 49y_N^{(4)}\right] - \frac{893}{60480}h^8y^{(8)}(\xi_N),$$

$$\xi_N \in (x_{N-2}, x_N). \qquad \ldots (3.7)$$

The derivation of these formulas can be effected by standard methods described by Fox (1957), or Henrici (1962).

Finally, our sixth order finite difference method is based on the scheme

$$\text{(i)} \quad -\frac{11}{2}y_0 + 9y_1 - \frac{9}{2}y_2 + y_3 = 3hy_0' - h^4\sum_{n=0}^{5}\beta_n y_n^{(4)} - \frac{1}{960}h^{10}y^{10}(\xi_1),\ \xi_1 \in (x_0, x_3);$$

$$\text{(ii)} \quad \delta^4 y_n = \frac{h^4}{720}\left[-y_{n-2}^{(4)} + 124y_{n-1}^{(4)} + 474y_n^{(4)} + 124y_{n+1}^{(4)} - y_{n+2}^{(4)}\right] + \frac{h^{10}}{3024}y^{(10)}(\xi_n),\ \xi_n \in (x_{n-2}, x_{n+2}),\ n = 2(1)N-2;$$

$$\text{(iii)} \quad y_{N-3} - 4y_{N-2} + 5y_{N-1} - 2y_N = -h^2y_N'' + h^4\sum_{i=0}^{5}\bar{\beta}_{5-i}y_{N-5+i}^{(4)} - \frac{501}{151200}y^{(10)}(\xi_{N-1}),\ \xi_{N-1} \in (x_{N-3}, x_n);$$

$$\text{(iv)} \quad y_{N-2} - 2y_{N-1} + y_N = h^2y_N'' - h^3y_N''' + \frac{h^4}{60480}\left[653y_{N-5}^{(4)} - 4158y_{N-4}^{(4)} + 11446y_{N-3}^{(4)} - 18272y_{N-2}^{(4)} + 30693y_{N-1}^{(4)} + 14918y_N^{(4)}\right]$$

*(equation continued on p. 402)*

$$- \frac{5071}{604800} h^{10} y^{10}(\xi_N), \xi_N \in (x_{N-5}, x_N). \qquad \ldots(3.8)$$

The coefficients $\beta_i$ and $\bar{\beta}_i$ in (3.8) are given by

$$(\beta_0, \ldots, \beta_5) = \frac{1}{33600}(937, 18240, 5990, 140, -135, 28) \qquad \ldots(3.9)$$

and

$$(\bar{\beta}_0, \ldots, \bar{\beta}_5) = \frac{1}{60480}(4233, 43274, 5662, 3432, -1391, 230). \qquad \ldots(3.10)$$

The standard details of the derivation of the preceding formulas are omitted for brevity.

## 4. Convergence of our Numerical Procedures

Let $Y = (y_n)$, $Z = (z_n)$, $C = (c_n)$, $T = (t_n)$, $E = (e_n)$ be $N$-dimensional column vectors with the error of discretization $e_n$ defined by

$$e_n = y_n - z_n \qquad \ldots(4.1)$$

where $y_n \equiv y(x_n)$ and $z_n$ is its numerical approximation.

We can now write the standard matrix equations for any of the three numerical methods of the previous section as follows :

$$\begin{aligned} &\text{(i)} \quad MY = C + T \\ &\text{(ii)} \quad MZ = C \\ &\text{(iii)} \quad ME = T. \end{aligned} \qquad \ldots(4.2)$$

We also have in each of the three cases

$$M = A + h^4 BF, \quad F = \operatorname{diag}(f_n) \qquad \ldots(4.3)$$

where $A$ is a nonsymmetric five band matrix given by

$$A = \begin{bmatrix} 9 & -\frac{9}{2} & 1 & & & & & \\ -4 & 6 & -4 & 1 & & & & \\ 1 & -4 & 6 & -4 & 1 & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & 1 & -4 & 6 & -4 & 1 \\ & & & & 1 & -4 & 5 & -2 \\ & & & & & 1 & -2 & 1 \end{bmatrix}. \qquad \ldots(4.4)$$

It is known (see Isa and Usmani 1980) that $A$ is a monotone matrix and if $A^{-1} = (a_{ij})$, then

$$\| A^{-1} \| = \max_m \sum_{n=1}^{N} a_{mn} = \frac{N}{24}(3N^3 + 4N^2 - N + 2)$$

(*equation continued on p.* 403)

$$= \frac{N^4}{24}\left[3 + \frac{4N^2 - N + 2}{N^3}\right] \approx \frac{N^4}{8} \qquad \ldots(4.5)$$

for large values of $N$, see Appendix $A$ for the proof.

Our main purpose now is to derive a bound on $\| E \| = \max_n | e_n |$. In order to accomplish this we need the following lemma.

*Lemma* 4.1—If $H$ is a square matrix of order $n$ with $\| H \| < 1$, then $(I + H)$ is nonsingular and

$$\| (I + H)^{-1} \| \leqslant \frac{1}{1 - \| H \|} .$$

For proof of Lemma 4.1, see Fröberg [1969, inequality (3.6.9)].

We now write the eqn. (4.2 iii) in the form

$$\begin{aligned} E &= M^{-1}T \\ &= (A + h^4BF)^{-1}T \\ &= (I + h^4A^{-1}BF)^{-1}A^{-1}T \end{aligned}$$

and

$$\| E \| \leqslant \frac{\| A^{-1} \| \cdot \| T \|}{1 - h^4 \| A^{-1} \| \cdot \| B \| \cdot \| F \|} , \quad \text{by Lemma 4.1} \qquad \ldots(4.6)$$

provided that

$$h^4 < 1/[\| A^{-1} \| \cdot \| B \| \cdot \| F \|]. \qquad \ldots(4.7)$$

For example, for our fourth finite difference method based on (3.7), we have

$$\| T \| < 0.0148\, h^8 M_8, \quad M_8 = \max_x | y^{(8)}(x) |$$

$$B = \begin{bmatrix} \frac{151}{280} & \frac{52}{280} & -\frac{1}{280} & & & & \\ \frac{1}{6} & \frac{4}{6} & \frac{1}{6} & & & & \\ & \frac{1}{6} & \frac{4}{6} & \frac{1}{6} & & & \\ & \cdots & \cdots & \cdots & & & \\ & & \frac{1}{6} & \frac{4}{6} & \frac{1}{6} & & \\ & & \frac{1}{360} & \frac{56}{360} & \frac{245}{360} & \frac{28}{360} & \\ & & \frac{4}{180} & -\frac{19}{180} & \frac{71}{180} & \frac{49}{180} & \end{bmatrix}, \qquad \ldots(4.8)$$

so that $\| B \| \leqslant 1$, $\| F \| = \max_x | f(x) | = f_M$

and

$$\| E \| \leqslant Kh^4, \qquad \ldots(4.9)$$

where

$$K = 0.0148M_8G[1 - f_MG]^{-1}$$

$$G = \frac{(b-a)^4}{8}\left[1 + \frac{4h}{3(b-a)} - \frac{h^2}{3(b-a)^2} + \frac{2h^3}{3(b-a)^3}\right]$$

$$= \frac{(b-a)^4}{8}[1 + O(h)]$$

provided that

$$f_M \leqslant G^{-1} \approx 8/(b-a)^4. \qquad \ldots(4.10)$$

While stronger results of the form (4.10) no doubt exist, the upper bound in (4.10) cannot be disregarded altogether. The inequality (4.9) proves the convergence of the method. In a similar manner we can prove the convergence of our remaining second and sixth order methods respectively. The matrix $B$ for our second and sixth order methods, and the vector $C$ introduced in (4.2) for all the three methods are given in Appendix B.

## 5. Shooting Method

A modified shooting technique, the details of which are briefly outlined below, can be used to obtain an approximate solution of the boundary value problem (1.1). We first show that the differential system (1.1) is equivalent to the following twelve linear first order differential equations. We set $y = y_1$ and

$$\begin{aligned}
&(1)\ y_1' = y_2, && y_1(a) = A_1\\
&(2)\ y_2' = y_3, && y_2(a) = A_2\\
&(3)\ y_3' = y_4, && y_3(a) = \alpha_0\\
&(4)\ y_4' = -f(x)\,y_1 + g(x), && y_4(a) = \beta_0\\
&(5)\ y_5' = y_6, && y_5(a) = 0\\
&(6)\ y_6' = y_7, && y_6(a) = 0\\
&(7)\ y_7' = y_8, && y_7(a) = 1\\
&(8)\ y_8' = -f(x)y_5, && y_8(a) = 0\\
&(9)\ y_9' = y_{10}, && y_9(a) = 0\\
&(10)\ y_{10}' = y_{11}, && y_{10}(a) = 0\\
&(11)\ y_{11}' = y_{12}, && y_{11}(a) = 0\\
&(12)\ y_{12}' = -f(x)y_9, && y_{12}(a) = 1. \qquad \ldots(5.1)
\end{aligned}$$

Here we write $\frac{\partial y_1}{\partial \alpha} = y_5$ and $\frac{\partial y_1}{\partial \beta} = y_9$, $\alpha_0$ and $\beta_0$ are parameters. In fact $\alpha_0$ and $\beta_0$ provide the initial guesses for $y''(a)$ and $y'''(a)$ respectively. The initial value problem (5.1) is solved by a fourth-order Runge-Kutta-method. The programming details for solving a set of $n$ first order differential equations are given by Ralston (1960, p. 110). The improved values of $y''(a)$ and $y'''(a)$ are determined by the iterative process

$$\alpha_{m+1} = \alpha_m + \frac{\Delta_1}{\Delta}$$

$$\beta_{m+1} = \beta_m + \frac{\Delta_2}{\Delta}, \quad (m = 0, 1, \ldots) \qquad \ldots(5.2)$$

where

$$\Delta_1 = y_{12}(b)\,[B_1 - y_3(b)] - y_{11}(b)\,[B_2 - y_4(b)]$$

$$\Delta_2 = y_7(b)\,[B_2 - y_4(b)] - y_8(b)\,[B_1 - y_3(b)]$$

$$\Delta = y_7(b)\,y_{12}(b) - y_8(b)\,y_{11}(b).$$

The iterative process (5.2) is terminated when

$$S = [[y_3(b) - B_1]^2 + [y_4(b) - B_2]^2]^{1/2} < \epsilon \qquad \ldots(5.3)$$

where $\epsilon$ is a preassigned small positive constant.

## 6. Survey of Numerical Experiments

In this section we illustrate the comparative usefulness of the three numerical techniques discussed in section 3 by solving the following two boundary value problems of the form (1.1):

$$y^{(4)}(x) + xy(x) = -(8 + 7x + x^3)e^x, f(x) \geqslant 0 \qquad \ldots(6.1)$$

$$\begin{cases} y(0) = 0, \; y'(0) = 1, \\ y''(1) = -4e, \; y'''(1) = -9e. \end{cases}$$

The exact solution of the preceding boundary value problem is

$$y(x) = x(1 - x)e^x \qquad \ldots(6.2)$$

which is used to check the accuracy of the numerical solutions obtained by finite difference methods and shooting technique.

$$y^{(4)}(x) - xy(x) = -(11 + 9x + x^2 - x^3)e^x. \qquad \ldots(6.3)$$

$$\begin{cases} y(-1) = 0, \; y'(-1) = \frac{2}{e} \\ y''(1) = -6e, \; y'''(1) = -12e. \end{cases}$$

The exact solution of this boundary value problem is

$$y(x) = (1 - x^2)e^x. \qquad \ldots(6.4)$$

All computations were done in double precision arithmetic using an Amdahl 470/7 computer at the University of Manitoba. The numerical experiments are briefly summarized in Tables I–III.

TABLE I

*Observed* $\| E \|$ *in the solution of the system* (6.1) *for* $h = 2^{-m}$, $m = 2(1)8$, $f(x) \geqslant 0$

| $m$ | 2nd order method | 4th order method | 6th order method |
|---|---|---|---|
| 2 | 0.850−1* | 0.637−3 | — |
| 3 | 0.209−1 | 0.270−4 | 0.311−6 |
| 4 | 0.527−2 | 0.109−5 | 0.274−8 |
| 5 | 0.133−2 | 0.454−7 | 0.213−10 |
| 6 | 0.333−3 | 0.209−8 | 0.645−11 |
| 7 | 0.836−4 | 0.234−9 | 0.688−10 |

* we write 0.850−1 for $0.850 \times 10^{-1}$.

TABLE II

*Observed* $\| E \|$ *for the problem* (6.3), $f(x)$ *changes sign,* $h = 2^{-m}$

| $m$ | 4th order method | 6th order method |
|---|---|---|
| 1 | 0.664 | — |
| 2 | 0.334−1 | — |
| 3 | 0.143−2 | 0.135−2 |
| 4 | 0.603−4 | 0.140−4 |
| 5 | 0.269−5 | 0.116−6 |
| 6 | 0.161−6 | 0.405−9 |

TABLE III

*Observed* $\| E \|$ *for the problem* (6.1), $\alpha_0 = -1$, $\beta_0 = -1$

| $h$ | Runge-Kutta 4th order method for modified shooting technique | Our 4th order finite difference method |
|---|---|---|
| 1/8 | 0.463−4 | 0.270−4 |
| 1/16 | 0.298−5 | 0.109−5 |
| 1/32 | 0.188−6 | 0.457−7 |
| 1/64 | 0.119−7 | 0.209−8 |
| 1/128 | 0.743−9 | 0.234−9 |

The present fourth-order finite difference method based on (3.7) gives numerical results that are marginally more accurate than those obtained by shooting technique employing Runge-Kutta method of the same order (Table III). Similar observations are made with other test problems not presented in this report.

Finally, we tabulate in Table IV the observed errors $e_n$ based on the 4th and sixth order finite difference methods given by (3.7) and (3.8) respectively at $x = 0(0.125)1$ for the problem (6.1).

TABLE IV

*Observed* $e_n = y_n - z_n$ *for the problem* (6.1), $h = 1/8$

| $x_i$ | Problem (6.1), $e_i$ in units of $10^{-5}$ | Problem (6.3), $e_i$ in units of $10^{-7}$ |
|---|---|---|
| 0.0 | 0 | 0.0 |
| 0.125 | 0.063 | 0.069 |
| 0.250 | 0.240 | 0.266 |
| 0.375 | 0.513 | 0.573 |
| 0.500 | 0.864 | 0.971 |
| 0.625 | 1.274 | 1.443 |
| 0.750 | 1.728 | 1.969 |
| 0.875 | 2.210 | 2.532 |
| 1. | 2.704 | 3.109 |

The numerical results can be further improved on employing Richardson's $h$-extrapolation approach.

Finally, smooth approximation of $y(x)$ satisfying (1.1) by the use of quintic spline function is recently being studied by Isa and Usmani (1980). The advantages of this approach are that it approximates $y(x)$ by a different spline function in each subinterval, thus providing approximations not only for $y(x)$, but also for the derivatives of $y(x)$ at every point of the interval $[a, b]$.

## REFERENCES

Fox, L. (1957). The Numerical Solution of Two-Point Boundary Value Problems in Ordinary Differential Equations. Oxford University Press, London.

Fröberg, Carl-Erik (1969). Introduction to Numerical Analysis, 2nd ed. Addison-Wesley, New York.

Henrici, P. (1962). Discrete Variable Methods in Ordinary Differential Equations. Willey, New York.

Isa, M., and Usmani, R. A. (1980). Quintic spline solutions of a boundary value problem. (unpublished report).

Kreiss, Heinz-Otto (1972). Difference approximations for boundary and eigenvalue problems for ordinary differential equations. *Math. Computation*, **26**, 605–24.

Ralston, J. W. (1960). Mathematical Methods for Digital Computers. Wiley, New York.

Reiss, E. L., Callegari, A. J., and Ahluwalia, D. S. (1976). Ordinary Differential Equations with Applications. Holt, Rinehart, and Winston, New York.

Steinberg, J. (1972). Numerical solution of a boundary value problem arising in the deflection of beams and shells. *J. Inst. Math. Applics.*, **10**, 10–14.

Usmani, R. A. (1978). Discrete variable methods for a boundary value problem with engineering applications. *Math. Computations*, **32**, 1087–96.

——— (1979). Discrete methods for boundary value problems with applications in plate deflection theory. *J. appl. Math, Phys., (ZAMP)*, **30**, 87–99.

——— (1980). Smooth spline approximations for the solution of a boundary value problem with engineering applications. *J. Comput. appl. Math.*, **6**.

Usmani, Riaz A., and Warsi, Shaukat, A. (1980). Smooth spline solutions for boundary value problems in plate deflection theory. *Computers Math. Applic., (CAMWA)*, **6**, 205–11.

## Appendix A

The formula (4.5) giving $\| A^{-1} \|$ is proved by Isa and Usmani (1980), but for the sake of completeness we shall outline the proof very briefly below.

We rely entirely on the theory of the linear difference equations with constant coefficients for generating explicitly the elements of the inverse of the five-band $(N \times N)$ matrix $A$ given by (4.4). Set

$$A^{-1} = (a_{ij}),\ \Delta_{i,j} = a_{i,j-2} - 4a_{i,j-1} + 6a_{i,j} - 4a_{i,j+1} + a_{i,j+2} \qquad \ldots(A.1)$$

and multiply the $i$th row $R_i$ of $A^{-1}$, namely

$$R_i = [a_{i1}\ a_{i2}\ a_{i3}\ \ldots\ a_{ij}\ \ldots\ a_{iN}] \qquad \ldots(A.2)$$

with the respective columns of $A$ to obtain the following fourth-order linear difference equations :

$$\begin{array}{ll} \text{(i)} & \left\{ \begin{array}{l} 0,\ j = 3(1)\,i - 1, \text{ with } 9a_{i_1} - 4a_{i_2} + a_{i_3} = 0 \\ -\dfrac{9}{2}\, a_{i_1} + 6a_{i_2} - 4a_{i_3} + a_{i_4} = 0\,; \end{array} \right. \\ \text{(ii) } \Delta_{ij} = & \left\{ \begin{array}{l} 1,\ j = i\,; \end{array} \right. \\ \text{(iii)} & \left\{ \begin{array}{l} 0,\ j = i + 1(1)N - 2,\ a_{i,N-3} - 4a_{i,N-2} + 5a_{i,N-1} - 2a_{iN} = 0\,, \\ a_{i,N-2} - 2a_{i,N-1} + a_{iN} = 0. \end{array} \right. \end{array} \qquad \ldots(A.3)$$

The solution of the difference equation (A.3i) with the corresponding initial conditions can easily be obtained in the form

$$a_{ij} = (1 - 3j^2)\,\bar{A} + (j - j^3)\bar{B},\ \ j = 1(1)i + 1 \qquad \ldots(A.4)$$

where the coefficients $\bar{A}$ and $\bar{B}$ are independent of $j$, but may be functions of $i$ alone. Similarly, the solution of (A.3 iii) with the associated conditions is

$$a_{ij} = \bar{C} + j\bar{D},\ \ j = i - 1(1)N \qquad \ldots(A.5)$$

and, $\bar{C}$ and $\bar{D}$ are functions of $i$ alone. Note that the elements $a_{ij}$, $j = i - 1\,(1)\,i+1$, can be obtained either from (A.4) or from (A.5). On equating the expressions for

these elements obtained from (A.4) and (A.5) respectively, we obtain three equations in the unknowns $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$. The fourth equation is obtained on substituting in (A.3 ii) the values of $a_{ij}$, $j = i - 2(1)i + 2$, derived either from (A.4) or (A.5). Finally, the unknowns $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$ are determined from the linear equations

$$\begin{bmatrix} \{1-3(i-1)^2\} & i(i-1)(i-2) & 1 & i-1 \\ \{i-3i^2\} & i(i-1)(i+1) & 1 & i-2 \\ \{1-3(i+1)^2\} & i(i+1)(i+2) & 1 & i+1 \\ -\{1-3(i-2)^2\} & (i-1)(i-2)(i-3) & 1 & i-2 \end{bmatrix} \begin{bmatrix} \bar{A} \\ \bar{B} \\ \bar{C} \\ \bar{D} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}. \qquad \ldots(A.6)$$

The preceding system of linear equations is easily solved by Cramer's rule or otherwise, and we have

$$(\bar{A}, \bar{B}, \bar{C}, \bar{D},) = \frac{1}{6}(-i, 1, -i^3, i^2). \qquad \ldots(A.7)$$

On substituting the values $\bar{A}$, $\bar{B}$, $\bar{C}$, and $\bar{D}$ in (A.5) and (A.6) respectively, we obtain

$$a_{ij} = \begin{cases} -\frac{i}{6}(1-3j^2) + \frac{1}{6}(j-j^3), & j = 1(1)\, i+1\,; \\ -\frac{i^3}{6} + \frac{i^2}{6}\, j, & j = i-1(1)N. \end{cases} \qquad \ldots(A\ 8)$$

It is easily proved that $S$ is a monotone matrix. Thus

$$a_{ij} > 0 \ \forall i, j = 1(1)N. \qquad \ldots(A.9)$$

In order to find $\| A^{-1} \|$, we first find the sum of the elements in modulus in the $i$th row of $A^{-1}$. Let us designate

$S_i = \sum_{j=1}^{N} |a_{ij}| = \sum_{j=1}^{N} a_{ij}$ by (A.9). We thus have

$$\begin{aligned} S_i &= \sum_{j=1}^{i} a_{ij} + \sum_{j=i+1}^{N} a_{ij} \\ &= \frac{i}{24}[i^3 - 2(2N+1)i^2 + (6N^2 + 6N - 1)\, i + 2]. \end{aligned} \qquad \ldots(A.10)$$

We can easily prove that the sequence $\{S_i\}$ is a monotonic increasing sequence, and thus $S_i$ is maximum for $i = N$.

Hence

$$\begin{aligned} \| A^{-1} \| &= S_N \\ &= \frac{N}{24}[3N^3 + 4N^2 - N + 2] \\ &\approx \frac{N^4}{8} = \frac{(b-a)^4}{8} h^{-4} \end{aligned} \qquad \ldots(A.11)$$

for large values of $N$. The preceding equality shows that $\| A^{-1} \| = O(h^{-4})$.

## APPENDIX B

We introduced three finite difference methods in section 4 for approximating the solution of the two-point boundary value problem (1.1). In each case the unknown vector $Z = (z_n)$, $n = 1(1)N$, is obtained by solving the system of linear equations $MZ = C$, [see (4.2 ii)], where $M = A + h^4BF$, $F = \text{diag}\,(f_n)$, [see (4.2) and (4.3)]. We give below the matrix $B$ and the vector $C$ for these methods to facilitate the user of these methods in practical computations.

For the second-order method defined by eqns. (3.4) and (3.6)

$$B = \begin{bmatrix} 1 & 0 & 0 & \cdots & & & \\ 0 & 1 & 0 & \cdots & & & \\ 0 & 0 & 1 & \cdots & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & \cdots & 0 & 1 & 0 & 0 & \\ 0 & \cdots & 0 & 0 & 1 & -\frac{1}{12} & \\ 0 & \cdots & 0 & 0 & \frac{1}{4} & \frac{1}{3} & \end{bmatrix},$$

and for $C = (c_n)$, we have

$$c_1 = \left(\frac{11}{2} + \frac{3}{20}\,h^4 f_0\right) A_1 + 3hA_2 + \frac{h^4}{20}\,(-3g_0 + 18g_1)$$

$$c_2 = h^4 g_2 - A_1$$

$$c_i = h^2 g_i, \quad i = 3(1)N-2$$

$$c_{N-1} = -h^2 B_1 + h^4\left(g_{N-1} - \frac{1}{12}\,g_N\right)$$

$$c_N = h^2 B_1 - h^3 B_2 + h^4\left(\frac{1}{4}\,g_{N-1} + \frac{1}{3}\,g_N\right).$$

For our fourth order method based on the formulas (3.7), we have already written down $B$ given by (4.8), and for the vector $C$, we have

$$c_1 = \left(\frac{11}{2} - \frac{8}{280}\,h^4 f_0\right) A_1 + 3hA_2 + \frac{h^4}{280}\,[8g_0 + 151g_1 + 52g_2 - g_3],$$

$$c_2 = -A_1 + \frac{h^4}{6}\,[g_1 + 4g_2 + g_3],$$

$$c_i = \frac{h^4}{6}\,[g_{i-1} + 4g_i + g_{i+1}], \quad i = 3(1)N-2,$$

$$c_{N-1} = -h^2 B_1 + \frac{h^4}{360}\,[g_{N-3} + 56g_{N-2} + 245g_{N-1} + 28g_N],$$

$$c_N = h^2 B_1 - h^3 B_2 + \frac{h^4}{180}\,[4g_{N-3} - 19g_{N-2} + 71g_{N-1} + 49g_N].$$

Finally, for our sixth-order method based on (3.8), we have

$$B = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & & & & & \\ b_1 & b_2 & b_3 & b_4 & & & & & & \\ b_0 & b_1 & b_2 & b_3 & b_4 & & & & & \\ & b_0 & b_1 & b_2 & b_3 & b_4 & & & & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & & & \\ & & & & b_0 & b_1 & b_2 & b_3 & b_4 & \\ & & & \bar{\beta}_5 & \bar{\beta}_4 & \bar{\beta}_3 & \bar{\beta}_2 & \bar{\beta}_1 & \bar{\beta}_0 & \\ & & & \gamma_5 & \gamma_4 & \gamma_3 & \gamma_2 & \gamma_1 & \gamma_0 \end{bmatrix},$$

and $(\beta_0,\ldots,\beta_5)$, $(\bar{\beta}_0,\ldots,\bar{\beta}_5)$ are given by (3.9) and (3.10) respectively, and

$$(b_0,\ldots,b_4) = \frac{1}{720}(-1,\ 124,\ 474,\ 124,\ -1),$$

$$(\gamma_0,\ldots,\gamma_5) = \frac{1}{60480}(14918,\ 30693,\ -18272,\ 11446,\ -4158,\ 653),$$

and

$$c_1 = \left(\frac{11}{2} - h^4\beta_0 f_0\right) A_1 + 3hA_2 + h^4 \sum_{i=0}^{5} \beta_i g_i$$

$$c_2 = \left(-1 + \frac{h^4}{720} f_0\right) A_1 + h^4 \sum_{i=0}^{4} b_i g_i$$

$$c_i = h^4(b_0 g_{i-2} + b_1 g_{i-1} + b_2 g_i + b_3 g_{i+1} + b_4 g_{i+2}),\ i = 3(1)N-2$$

$$c_{N-1} = -h^2 B_1 + h^4 \sum_{i=0}^{5} \bar{\beta}_{5-i}\, g_{N-5+i},$$

$$c_N = h^2 B_1 - h^3 B_2 + h^4 \sum_{i=0}^{5} \gamma_{5-i}\, g_{N-5+i}.$$