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SOME INTEGRAL INEQUALITIES FOR POLYNOMIALS

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In this paper we have obtained L^2 inequalities for the sth derivative of the classes of polynomials having no zeros in $|z| < K, K \leqslant 1$ and for polynomials having no zeros in $|z| < K, K \geqslant 1$. The inequality for the case $K \leqslant 1$ is best possible. Our results generalize some well-known results.

§ 1. Let $p(z) = \sum_{v=0}^{n} a_v z^v$ be a polynomial of degree n, then clearly

$$\int_0^{2\pi} |p'(e^{i\theta})|^2 d\theta = 2\pi \sum_{v=0}^{n} |v a_v|^2 \leqslant n^2 \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta. \quad \ldots(1.1)$$

If $p(z) \neq 0$ in $|z| < 1$, then the inequality (1.1) can be replaced by

$$\int_0^{2\pi} |p'(e^{i\theta})|^2 d\theta \leqslant n^2/2 \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta. \quad \ldots(1.2)$$

Inequality (1.2) is due to Lax (1944) [for another proof see de Bruijn (1947)] Rahman (1964) considered the class of polynomials not vanishing in $|z| < K, K \leqslant 1$ and proved

Theorem A—If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n, $p(z) \neq 0$ for $|z| < K, K \leqslant 1$ then

$$\int_0^{2\pi} |p'(e^{i\theta})|^2 d\theta \leqslant \frac{n^2}{1 + K^{2n}} \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta. \quad \ldots(1.3)$$

The result is best possible with equality for the polynomial $p(z) = \alpha z^n + \beta K^n$, $|\alpha| = |\beta|$.

For polynomials not vanishing in $|z| < K, K \geqslant 1$ it has been proved by Govil and Rahman [(1969), inequality (1.18), p. 505] that

$$\int_0^{2\pi} |p'(e^{i\theta})|^2 d\theta \leqslant \frac{n^2}{1 + K^2} \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta. \quad \ldots(1.4)$$

We do not know if the inequality (1.4) is sharp.

It is clearly of interest to obtain inequalities analogous to (1.3) and (1.4) for sth derivative.

If $p(z)$ is a polynomial of degree n, $p(z) \neq 0$ in $|z| < K$, $K \leqslant 1$ then by Theorem A (see Rahman 1964).

$$\int_0^{2\pi} |p'(e^{i\theta})|^2 \, d\theta \leqslant \frac{n^2}{1+K^{2n}} \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta.$$

Now $p'(z)$ is a polynomial of degree $(n-1)$, hence if we apply inequality (1.1) to to $p'(z)$ and combine it with the above inequality, we easily get

$$\int_0^{2\pi} |p''(e^{i\theta})|^2 \, d\theta \leqslant \frac{n^2(n-1)^2}{1+K^{2n}} \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta \qquad \text{...(1.5)}$$

and using principle of Mathematical Induction, inequality (1.5) yields

$$\int_0^{2\pi} |p^{(s)}(e^{i\theta})|^2 \, d\theta \leqslant \frac{n^2(n-1)^2 \ldots (n-s+1)^2}{1+K^{2n}} \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta. \qquad \text{...(1.6)}$$

Inequality (1.6) is best possible with equality holding for $p(z) = \alpha z^n + \beta K^n$, $|\alpha| = |\beta|$.

For $s = 1$, inequality (1.6) reduces to the inequality (1.3) due to Rahman (1964), and for $s = 1$, $K = 1$, the inequality (1.6) reduces to the inequality (1.2) due to Lax (1944) and de Bruijn (1947).

If one uses a similar argument to the case when $p(z)$ has no zeros in $|z| < K$, $K \geqslant 1$, one gets only

$$\int_0^{2\pi} |p^{(s)}(e^{i\theta})|^2 \, d\theta \leqslant \frac{n^2(n-1)^2 \ldots (n-s+1)^2}{1+K^2} \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta. \qquad \text{...(1.7)}$$

In this paper, we obtain a bound which is much better than in (1.7). More precisely we prove

Theorem—If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n, $p(z) \neq 0$ in

$|z| < K$, $K \geqslant 1$, then

$$\int_0^{2\pi} |p^{(s)}(e^{i\theta})|^2 \, d\theta \leqslant \frac{n^2(n-1)^2 \ldots (n-s+1)^2}{1+K^{2s}} \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta. \qquad \text{...(1.8)}$$

Inequality (1.8) is obviously a generalization of (1.4).

§ 2. *Lemma* 1—If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n, not vanishing in $|z| < K$, $K \geqslant 1$, then

$$K^s |p^{(s)}(e^{i\theta})| \leqslant |q^{(s)}(e^{i\theta})|, \; 0 \leqslant \theta < 2\pi \qquad \ldots(2.1)$$

where

$$q(z) = z^n \overline{p(1/\bar{z})}.$$

This result is due to Govil and Rahman [for proof see Govil and Rahman (1969, inequality (3.3), p. 511)].

Lemma 2—Let v and s be integers such that $0 \leqslant v \leqslant n$ and $1 \leqslant s \leqslant n$. Then we have

$$v^2 (v-1)^2 \ldots (v-s+1)^2 + (n-v)^2 (n-v-1)^2 \ldots (n-v-s+1)^2$$
$$\leqslant n^2 (n-1)^2 \ldots (n-s+1)^2.$$

PROOF OF LEMMA 2 : The result trivially holds for $s = 1$ and it is easy to verify that the result holds true for $s = k+1$ whenever it is true for $s = k$, hence Lemma 2 follows from the principle of Mathematical Induction.

§ 3. *Proof of the Theorem*—Since for each θ, $0 \leqslant \theta < 2\pi$

$$|p^{(s)}(e^{i\theta})|^2 = \frac{1}{1+K^{2s}} \left(|p^{(s)}(e^{i\theta})|^2 + K^{2s} |p^{(s)}(e^{i\theta})|^2 \right).$$

we find by using Lemma 1

$$|p^{(s)}(e^{i\theta})|^2 \leqslant \frac{1}{1+K^{2s}} \left(|p^{(s)}(e^{i\theta})|^2 + |q^{(s)}(e^{i\theta})|^2 \right).$$

Integrating both the sides with respect to θ from 0 to 2π, we get

$$\int_0^{2\pi} |p^{(s)}(e^{i\theta})|^2 d\theta \leqslant \frac{1}{1+K^{2s}} \left(\int_0^{2\pi} |p^{(s)}(e^{i\theta})|^2 d\theta + \int_0^{2\pi} |q^{(s)}(e^{i\theta})|^2 d\theta \right)$$

$$= \frac{2\pi}{1+K^{2s}} \left\{ \sum_{v=0}^{n} v^2 (v-1)^2 \ldots (v-s+1)^2 |a_v|^2 + \sum_{v=0}^{n} (n-v)^2 (n-v-1)^2 \ldots (n-v-s+1)^2 |\bar{a}_v|^2 \right\}$$

$$= \frac{2\pi}{1+K^{2s}} \sum_{v=0}^{n} \{ v^2 (v-1)^2 \ldots (v-s+1)^2 + (n-v)^2 (n-v-1)^2 \ldots (n-v-s+1)^2 \} |a_v|^2$$

which on using Lemma 2, gives

$$\int_0^{2\pi} | p^{(s)} (e^{i\theta}) |^2 d\theta \leqslant \frac{2\pi}{1 + K^{2s}} n^2 (n - 1)^2 \dots (n - s + 1)^2 \sum_{\nu=0}^{n} | a_\nu |^2$$

$$= \frac{n^2 (n - 1)^2 \dots (n - s + 1)^2}{1+K^{2s}} \int_0^{2\pi} | p (e^{i\theta}) |^2 d\theta$$

and the proof of the theorem is complete.

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