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CURVATURE COLLINEATION AND MAXWELL COLLINEATION IN SOME CYLINDRICALLY SYMMETRIC ELECTROVAC UNIVERSES

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In this paper are investigated the conditions under which the cylindrically symmetric electrovac universes given by Singh and Abdussattar admit curvature collineation and Maxwell collineation.

1. INTRODUCTION

Recently Singh and Abdussattar (1977) have obtained a solution for certain special cases of the most general cylindrically symmetric metric of Marder satisfying the vacuum field equations of electromagnetism. In this paper we investigate the conditions under which these cylindrically symmetric electrovac universes admit curvature collineation and Maxwell collineation respectively.

The metrics under consideration are given by Singh and Abdussattar (1977),

$$\text{(a)}\quad ds^2 = dt^2 - dx^2 - \{\cos(kt \pm px)\}^2\, dy^2 - dz^2 \qquad \ldots(1.1)$$

$$\text{(b)}\quad ds^2 = dt^2 - dx^2 - \{\cos px\}^2\, dy^2 - \{\cos kt\}^2\, dz^2. \qquad \ldots(1.2)$$

For the above two space-times the non-vanishing components of electromagnetic field tensors $F_{ij,t}$ are

$$\left.\begin{aligned} F_{12} &= -\frac{1}{\sqrt{8\pi}}\ (\pm p)\cos(kt \pm px) \\ F_{24} &= \frac{1}{\sqrt{8\pi}}\ k\cos(kt \pm px) \end{aligned}\right\} \qquad \ldots(1.3)$$

and

$$\left.\begin{aligned} F_{12} &= -\frac{p}{\sqrt{8\pi}}\ \cos(px) \\ F_{34} &= \frac{k}{\sqrt{8\pi}}\ \cos(kt) \end{aligned}\right\} \qquad \ldots(1.4)$$

respectively.

2. CURVATURE COLLINEATION

A space-time is said to admit curvature collineation, Katzin *et al.* (1969) along a field vector ξ^i if

$$\mathcal{L}_\xi\, R^h_{ijk} = 0 \qquad \ldots(2.1)$$

where $\mathcal{L}_\xi$ stands for the Lie derivative along the field vector ξ^i.

Case (*a*)—The non-vanishing components of the Riemann curvature tensor for the metric (1.1) are

$$\left.\begin{aligned} R^1_{212} &= \cos^2(kt \pm px)\, p^2, \\ R^2_{121} &= p^2,\ R^2_{424} = k^2, \\ R^4_{242} &= -\cos^2(kt \pm px)\, k^2, \\ R^1_{224} &= -\cos^2(kt \pm px)(\pm p)\, k, \\ R^2_{142} &= (\mp p)\, k. \end{aligned}\right\} \qquad \ldots(2.2)$$

Only six, of the curvature components are non-vanishing. But for CC we need all the components because they give rise to different equations in (2.1). It may be noted that for the vanishing components of Riemann tensor the Lie derivative is not necessarily zero.

Introducing the result (2.2) in (2.1) we get for the vanishing components of Riemann curvature tensor the equations

$$\left.\begin{aligned} &\mathcal{L}_\xi R^2_{321} = 0 \Longrightarrow \xi^1_{;3} = 0, \\ &\mathcal{L}_\xi R^2_{414} = 0 \Longrightarrow \xi^2_{;1} = 0, \\ &\mathcal{L}_\xi R^2_{141} = 0 \Longrightarrow p\,\xi^2_{;4} \pm k\,\xi^2_{;1} = 0, \\ &\mathcal{L}_\xi R^1_{142} = 0 \Longrightarrow \cos^2(kt \pm px)\,\xi^2_{;1} + \xi^1_{;2} = 0, \\ &\mathcal{L}_\xi R^4_{424} = 0 \Longrightarrow \cos^2(kt \pm px)\,\xi^2_{;4} + \xi^4_{;2} = 0, \\ &\mathcal{L}_\xi R^3_{423} = 0 \Longrightarrow \xi^4_{;3} = 0, \\ &\mathcal{L}_\xi R^3_{424} = 0 \Longrightarrow \xi^3_{;1} = 0, \\ &\mathcal{L}_\xi R^3_{242} = 0 \Longrightarrow \xi^3_{;4} = 0, \\ &\mathcal{L}_\xi R^3_{414} = 0 \Longrightarrow \xi^3_{;1} = 0, \\ &\mathcal{L}_\xi R^4_{342} = 0 \Longrightarrow \xi^2_{;3} = 0. \end{aligned}\right\} \qquad \ldots(2.3)$$

respectively $p,\ k \neq 0$.

By inspection of the above set of equations we find that

$$\left.\begin{array}{llllll} \text{(i)} & \xi^4_{,2} = 0 & & \xi^4_{,3} = 0 & \text{implying} & \xi^4 = \xi^4(x, t), \\ \text{(ii)} & \xi^1_{,2} = 0 & & \xi^1_{,3} = 0 & \text{implying} & \xi^1 = \xi^1(x, t), \\ \text{(iii)} & \xi^2_{,1} = 0 & \xi^2_{,3} = 0 & \xi^2_{,4} = 0 & \text{implying} & \xi^2 = \xi^2(y), \\ \text{(iv)} & \xi^3_{,1} = 0 & \xi^3_{,2} = 0 & \xi^3_{,4} = 0 & \text{implying} & \xi^3 = \xi^3(z). \end{array}\right\} \quad \ldots(2.4)$$

For the non-vanishing components of the Riemann curvature tensor, eqns. (2.1) give

$$\left.\begin{array}{l} \mathcal{L}_\xi R^1_{212} = 0 \Longrightarrow (\pm p)\,\xi^1 - k\xi^4 + \cot(kt \pm px)\, p\xi^2_{,2} = 0, \\ \mathcal{L}_\xi R^2_{121} = 0 \Longrightarrow \xi^1_{,1} = 0, \\ \mathcal{L}_\xi R^2_{424} = 0 \Longrightarrow \xi^4_{,4} = 0, \\ \mathcal{L}_\xi R^4_{242} = 0 \Longrightarrow (\pm p)\,\xi^1 + k\xi^4 - \cot(kt \pm px)\, k\xi^2_{,2} = 0, \\ \mathcal{L}_\xi R^1_{224} = 0 \Longrightarrow (p\xi^1 + (\pm p)\, k\xi^4 - (\pm 1)\cot \\ \qquad\qquad (kt \pm px)\,\xi^2_{,2} = 0, \\ \mathcal{L}_\xi R^2_{142} = 0 \Longrightarrow \xi^2_{,2} = 0. \end{array}\right\} \quad \ldots(25)$$

The above sets of equations are consistent if,

$$\xi^1 = \text{constant and } \xi^4 = \text{constant.} \quad \ldots(2.6)$$

Hence, we get

$$k\xi^4 \pm p\xi^1 = 0. \quad \ldots(2.7)$$

From (2.6) and (2.7)

$$\xi^1 = A \text{ (say)}, \quad \xi^4 = \pm \frac{p}{k} A. \quad \ldots(2.8)$$

From (2.5) and (2.4)

$$\xi^2 = B \text{ (say)}, \quad \xi^3 = \xi^3(y) \quad \ldots(2.9)$$

i.e.,

$$\xi^1 = A, \quad \xi^2 = B, \quad \xi^3 = \xi^3(y), \quad \xi^4 = \pm \frac{p}{k} A. \quad \ldots(2.10)$$

where A and B are constants.

Case (*b*)—The non-vanishing components of the Riemann curvature tensor for the metric (1.2) are

$$\left.\begin{aligned} R^1_{212} &= -p^2\cos^2 px\,, & R^4_{343} &= -k^2\cos^2 kt \\ R^2_{121} &= -p^2\,, & R^3_{434} &= -k^2\,. \end{aligned}\right\} \qquad ...(2.11)$$

Introducing (2.11) in (2.1), we get for the vanishing components of Riemann curvature tensor the equations.

$$\left.\begin{aligned} &\mathcal{L}_\xi R^2_{343} = 0 \Longrightarrow \xi^2_{,4} = 0\,, \\ &\mathcal{L}_\xi R^2_{434} = 0 \Longrightarrow \xi^2_{,3} = 0\,, \\ &\mathcal{L}_\xi R^4_{341} = 0 \Longrightarrow \xi^3_{,1} = 0\,, \\ &\mathcal{L}_\xi R^4_{342} = 0 \Longrightarrow \xi^3_{,2} = 0\,, \\ &\mathcal{L}_\xi R^2_{321} = 0 \Longrightarrow \xi^1_{,3} = 0\,, \\ &\mathcal{L}_\xi R^1_{343} = 0 \Longrightarrow \xi^4_{,4} = 0\,, \\ &\mathcal{L}_\xi R^4_{121} = 0 \Longrightarrow \xi^4_{,1} = 0\,, \\ &\mathcal{L}_\xi R^2_{212} = 0 \Longrightarrow \cos^2 px\xi^2_{,1} - \xi^1_{,2} = 0\,, \\ &\mathcal{L}_\xi R^4_{323} = 0 \Longrightarrow \xi^4_{,2} = 0\,, \\ &\mathcal{L}_\xi R^4_{434} = 0 \Longrightarrow \cos^2 kt\xi^3_{,4} + \xi^4_{,3} = 0\,. \end{aligned}\right\} \qquad ...(2.12)$$

By inspection of the above set of equations we find that

$$\left.\begin{aligned} &\text{(i)} \quad \xi^4_{,1} = 0 \quad \xi^4_{,2} = 0 \Longrightarrow \xi^4 = \xi^4(z, t)\,, \\ &\text{(ii)} \quad \xi^1_{,3} = 0 \quad \xi^1_{,4} = 0 \Longrightarrow \xi^1 = \xi^1(x, y)\,, \\ &\text{(iii)} \quad \xi^2_{,3} = 0 \quad \xi^2_{,4} = 0 \Longrightarrow \xi^2 = \xi^2(x, y)\,, \\ &\text{(iv)} \quad \xi^3_{,1} = 0 \quad \xi^3_{,2} = 0 \Longrightarrow \xi^3 = \xi^3(z, t)\,. \end{aligned}\right\} \qquad ...(2.13)$$

For the non-vanishing components of the Riemann curvature teesor (2.1) gives

$$\left.\begin{aligned} R^1_{212} &= 0 \Longrightarrow p \tan px\, \xi^1 - \xi^2_{,2} = 0\,, \\ R^2_{121} &= 0 \Longrightarrow \xi^1_{,1} = 0\,, \\ R^4_{343} &= 0 \Longrightarrow k \tan kt\, \xi^4 - \xi^3_{,3} = 0\,, \\ R^3_{434} &= 0 \Longrightarrow \xi^4_{,4} = 0\,, \end{aligned}\right\} \quad \ldots(2.14)$$

Hence, from eqns. (2.15) and (2.14) we infer that

$$\xi^1 = \xi^1(y),\ \xi^2 = \xi^2(x, y),\ \xi^3 = \xi^3(z, t),\ \xi^4 = \xi^4(z). \quad \ldots(2.15)$$

Also, the collineation vector in (2.10) and (2.15) satisfies the following equations.

$$\mathcal{L}_\xi g_{ij} \neq 0\,, \text{ and } \mathcal{L}_\xi \left\{ \begin{matrix} i \\ jR \end{matrix} \right\} \neq 0$$

which proves that the collineation is a proper CC.

3. Maxwell Collineation

Electromagnetic fields are said to admit Maxwell collineation (Collinson 1970) along the vector field ξ^i if

$$\mathcal{L}_\xi F^i_j = 0 \quad \ldots(3.1)$$

in which case the vector field ξ^i is called the MC vector.

Case (*a*)—Non-vanishing components of F_{ij} for the metric (1.1) are,

$$\left.\begin{aligned} F_{12} &= -\frac{1}{\sqrt{8\pi}} (\pm p) \cos(kt \pm px) \\ F_{24} &= \frac{1}{\sqrt{8\pi}} k \cos(kt \pm px)\,. \end{aligned}\right\} \quad (3.2)$$

Introducing (3.2) in (3.1), we get for the vanishing components of the electromagnetic field tensor equations :

$$\mathcal{L}_\xi F^1_1 = 0 \Longrightarrow \xi^1_{,2} + \cos^2(kt \pm px)\, \xi^2_{,1} = 0\,,$$

$$\mathcal{L}_\xi F^4_4 = 0 \Longrightarrow \xi^4_{,2} - \cos^2(kt \pm px)\, \xi^3_{,4} = 0\,,$$

$$\mathcal{L}_\xi F^1_3 = 0 \Longrightarrow \xi^2_{,3} = 0\,,$$

$$\mathcal{L}_\xi F^1_4 = 0 \Longrightarrow \xi^1_{,2} + \frac{(\pm p)}{k} \cos^2(kt \pm px)\, \xi^2_{,4} = 0\,,$$

$$\mathcal{L}_\xi F^2_3 = 0 \Longrightarrow \xi^1_{,3} + \frac{k}{\pm p} \xi^4_{,3} = 0\,,$$

$$F^3_4 = 0 \Longrightarrow \xi^3_{,2} = 0\,,$$

(*equation continued on p.* 10)

$$\left.\begin{aligned} F_1^4 = 0 \implies & \xi^4_{,3} - \frac{k}{\pm p} \cos^2 (kt \pm px)\ \xi^2_{,1} = 0 , \\ F_2^3 = 0 \implies & \xi^3_{,4} - \frac{(\pm p)}{k}\ \xi^3_{,1} = 0. \end{aligned}\right\} \quad ...(3.3)$$

For the above equations to be consistent

$$\left.\begin{aligned} & \xi^1_{,2} = \xi^1_{,3} = 0 , \quad \xi^2_{,1} = \xi^2_{,3} = \xi^2_{,4} = 0 \\ & \xi^3_{,1} = \xi^3_{,2} = \xi^3_{,4} = 0 , \quad \xi^4_{,2} = \xi^4_{,3} = 0 . \end{aligned}\right\} \quad ...(3.4)$$

Lie derivative of non-vanishing components of F^i_j gives

$$\left.\begin{aligned} \mathcal{L}_\xi F_2^1 = 0 \implies & (\pm p) \tan (kt \pm px)\, \xi^1 + k \tan (kt \pm px) \\ & \xi^4 + \xi^1_{,1} - \xi^2_{,2} - \frac{k}{\pm p}\ \xi^1_{,4} = 0 , \\ \mathcal{L}_\xi F_1^2 = 0 \implies & (\pm p) \tan (kt \pm px)\, \xi^1 + k \tan (kt \pm px) \\ & \xi^4 - \xi^2_{,2} + \xi^1_{,1} + \frac{\pm p}{k}\ \xi^4_{,1} = 0 , \\ \mathcal{L}_\xi F_4^2 = 0 \implies & (\pm p) \tan (kt \pm px)\, \xi^1 + k \tan (kt \pm px) \\ & \xi^4 - \xi^2_{,2} + \xi^4_{,4} + \frac{(\pm p)}{k}\ \xi^1_{,4} = 0 , \\ \mathcal{L}_\xi F_2^4 = 0 \implies & (\pm p) \tan (kt \pm px)\, \xi^1 + k \tan (kt \pm px) \\ & \xi^4 - \xi^2_{,2} + \xi^4_{,4} - \frac{(\pm p)}{k}\ \xi^4_{,1} = 0 . \end{aligned}\right\} \quad ...(3.5)$$

For the above equations to be consistent

$$\xi^1_{,1} = \xi^1_{,4} = \xi^4_{,1} = \xi^2_{,2} = 0,\ \xi^4_{,4} = 0 \quad ...(3.6)$$

and $\xi^1 = \pm\, k/p\ \xi^4$.

Hence, from equations (3.4) and (3.6), we have

$$\xi^1 = c_1 , \ \xi^2 = c_2\ \xi^3 = \xi^3 (y) , \ \xi^4 = \pm \frac{p\, c_1}{k} . \quad ...(3.7)$$

where c_1 and c_2 are arbitrary constants.

From the set of eqns. (2.10) and (3.7) we see that the space-time given by the line-element (1.1) admits curvature collineation and Maxwell collineation along the vector ξ^i.

Case (*b*)—The non-vanishing components of F_{ij} for the metric (1.2) are

$$\left.\begin{aligned} F_{12} &= -\frac{p}{\sqrt{8\pi}}\cos(px) \\ F_{34} &= \frac{k}{\sqrt{8\pi}}\cos(kt) \end{aligned}\right\} \qquad \ldots(3.8)$$

Lie derivative of the vanishing components of F^i_j gives ,

$$\left.\begin{aligned}
&\mathcal{L}_\xi F^1_1 = 0 \Longrightarrow \xi^1_{,2} - \cos^2 px\, \xi^2_{,1} = 0 , \\
&\mathcal{L}_\xi F^3_3 = 0 \Longrightarrow \xi^4_{,3} + \cos^2 kt\, \xi^3_{,4} = 0 , \\
&\mathcal{L}_\xi F^1_3 = 0 \Longrightarrow \xi^1_{,4} - \frac{p}{k}\frac{\cos px}{\cos kt}\xi^2_{,3} = 0 , \\
&\mathcal{L}_\xi F^2_4 = 0 \Longrightarrow \xi^1_{,4} - \frac{k}{p}\frac{\cos px}{\cos kt}\xi^2_{,3} = 0 , \\
&\mathcal{L}_\xi F^1_4 = 0 \Longrightarrow \xi^1_{,3} + \frac{p}{k}\cos px \cos kt\, \xi^2_{,4} = 0 , \\
&\mathcal{L}_\xi F^2_3 = 0 \Longrightarrow \xi^1_{,3} + \frac{k}{p}\cos px \cos kt\, \xi^2_{,4} = 0 , \\
&\mathcal{L}_\xi F^3_1 = 0 \Longrightarrow \xi^4_{,1} - \frac{k}{p}\frac{\cos kt}{\cos px}\xi^3_{,2} = 0 , \\
&\mathcal{L}_\xi F^4_2 = 0 \Longrightarrow \xi^4_{,1} - \frac{p}{k}\frac{\cos kt}{\cos px}\xi^3_{,2} = 0 , \\
&\mathcal{L}_\xi F^4_1 = 0 \Longrightarrow \xi^4_{,2} + \frac{k}{p}\cos px \cos kt\, \xi^3_{,1} = 0 , \\
&\mathcal{L}_\xi F^3_2 = 0 \Longrightarrow \xi^4_{,2} + \frac{p}{k}\cos px \cos kt\, \xi^3_{,1} = 0 .
\end{aligned}\right\} \qquad \ldots(3.9)$$

Since, $p \neq k$, we have from the above set of equations

$$\left.\begin{aligned}
&\text{(i)} \quad \xi^4_{,1} = 0 \quad \xi^4_{,2} = 0 \Longrightarrow \xi^4 = \xi^4(z, t), \\
&\text{(ii)} \quad \xi^1_{,3} = 0 \quad \xi^1_{,4} = 0 \Longrightarrow \xi^1 = \xi^1(x, y), \\
&\text{(iii)} \quad \xi^2_{,3} = 0 \quad \xi^2_{,4} = 0 \Longrightarrow \xi^2 = \xi^2(x, y), \\
&\text{(iv)} \quad \xi^3_{,1} = 0 \quad \xi^3_{,2} = 0 \Longrightarrow \xi^4 = \xi^4(z, t) .
\end{aligned}\right\} \qquad (3.10)$$

Lie derivative of the non-vanishing components of F^i_j gives

$$F^1_2 = 0 , F^4_3 = 0. \qquad \ldots(3.11)$$

Case (b_i)—If,

$$\xi^2_{,2} = \xi^3_{,3} = 0. \qquad \ldots(3.12)$$

Then from the set of eqns. (3.10), (3.11) and (3.12), we have

$$\xi^1 = c_3(y) \cos px, \ \xi^2 = \xi^2(x), \ \xi^3 = \xi^3(t), \ \xi^4 = c_4(z) \cos kt \qquad \ldots(3.13)$$

where $c(y)$ and $c(z)$ are some function of y and z.

Case (b_{ii})—If,

$$\xi^1_{,1} = \xi^4_{,4} = 0 \qquad \ldots(3.14)$$

then from the set of eqns. (3.10) and (3.11) we have

$$\xi^1 = \xi^1(y), \ \xi^2 = \xi^2(x, y) \qquad \ldots(3.15a)$$

$$\xi^3 = \xi^3(z, t), \ \xi^4 = \xi^4(z). \qquad \ldots(3.15b)$$

From eqns. (2.15) and (3.15a, b) we see that for the space-time given by the line-element (1.2), admits curvature collineation and Maxwell collineation along the same vector.

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