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SOME RESULTS ON PSEUDO-HEREDITARY TORSION THEORIES

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In this paper we give a method for obtaining pseudo-hereditary torsion theories and characterize the class of rings over which the only pseudo-hereditary torsion theories are the trivial ones.

1. PRELIMINARIES

Let R be a ring with identity and R-mod be the category of all left R-modules. By a torsion theory of R-mod we mean, a pair of subclasses $(\mathcal{T}, \mathcal{F})$ of R-mod which satisfies the following two conditions :

(1) $\text{Hom}(T, F) = 0 \ \forall \ T \in \mathcal{T}$ and $\forall \ F \in \mathcal{F}$.

(2) The classes $\mathcal{T}$ and $\mathcal{F}$ are maximal with respect to condition (1)

The class $\mathcal{T}$ is called the class of torsion modules whereas the class $\mathcal{F}$ is called the class of torsionfree modules.

A torsion theory $(\mathcal{T}, \mathcal{F})$ is called hereditary (cohereditary) if $\mathcal{T}$ ($\mathcal{F}$) is closed under taking submodules (factors).

If $(\mathcal{T}, \mathcal{F})$ is a torsion theory of R-mod, then for any R-module M there exists a unique largest submodule $T(M)$ of M which belongs to $\mathcal{T}$. The submodule $T(M)$ is called the torsion submodule of M.

Given a class $\mathcal{C}$ of R-modules, let

$$\mathcal{T} = \{T \in R\text{-mod} \mid \text{Hom}(T, C) = 0 \ \forall \ C \in \mathcal{C}\}$$

and

$$\mathcal{F} = \{F \in R\text{-mod} \mid \text{Hom}(T, F) = 0 \ \forall \ T \in \mathcal{T}\}.$$

Then $(\mathcal{T}, \mathcal{F})$ is a torsion theory of R-mod and is called the torsion theory cogenerated by the class $\mathcal{C}$.

A torsion theory $(\mathcal{T}, \mathcal{F})$ of R-mod is called stable if it is closed under injective hulls and is called a *TTF* theory if it is hereditary and $\mathcal{T}$ is closed under taking products.

For various properties of torsion theories we refer the reader to Stenstrom (1975).

2. Pseudo-Hereditary Torsion Theories

In this section we study some properties of pseudo-hereditary torsion theories. We begin with the definition of a pseudo-hereditary torsion theory.

Definition 2.1—A torsion theory $(\mathcal{T}, \mathcal{F})$ of R-mod is called pseudo-hereditary if every left $I \subseteq T(R)$ of R is in $\mathcal{T}$.

Remark 2.2 : Clearly every hereditary torsion theory is pseudo-hereditary and every faithful torsion theory (i. e. a torsion theory for which $T(R) = 0$) is pseudo-hereditary.

Example 2.3—Let Z be the ring of integers. Let $\mathcal{D}$ be the class of all divisible groups and $\mathcal{R}$ be the class of all reduced groups. Then $(\mathcal{D}, \mathcal{R})$ is a torsion theory of Z--mod. Since Z is reduced by the above remark, $(\mathcal{D}, \mathcal{R})$ is pseudo-hereditary but $(\mathcal{D}, \mathcal{R})$ is not hereditary.

Proposition 2.4—The following are equivalent for a torsion theory $(\mathcal{T}, \mathcal{F})$:

(1) $(\mathcal{T}, \mathcal{F})$ is pseudo-hereditary.

(2) For every left ideal I of R, $T(I) = I \cap T(R)$.

(3) For every R-module M, $N \subseteq T(R)$. M implies $N \in \mathcal{T}$.

Proof : (1) $\Leftrightarrow$ (2) Follows easily from the definition of pseudo-heredity.

(1) $\Leftrightarrow$ (3) (Rangaswamy 1974).

Corollary 2.5—A cohereditary torsion theory $(\mathcal{T}, \mathcal{F})$ is hereditary if and only if it is pseudo-hereditary.

Proof : Follows easily from condition (3) of the above proposition.

It is well known that a torsion theory $(\mathcal{T}, \mathcal{F})$ is hereditary if and only if it is cogenerated by a class of injective modules. Now we show that a torsion theory cogenerated by a class of absolutely pure modules is always pseudo-hereditary.

Proposition 2.6— Let $\mathcal{C}$ be a class of absolutely pure modules. Let $(\mathcal{T}, \mathcal{F})$ be the torsion theory cogenerated by $\mathcal{C}$. Then $(\mathcal{T}, \mathcal{F})$ is pseudo-hereditary.

Proof : It is enough to show that every cyclic submodule of $T(R)$ is in $\mathcal{T}$. Suppose $0 \neq Rx \subseteq T(R)$ such that $Rx \notin \mathcal{T}$. Since $\mathcal{C}$ cogenerates $(\mathcal{T}, \mathcal{F})$, there exists a $C \in \mathcal{C}$ and an $f \in \mathrm{Hom}(Rx, C)$ such that $f \neq 0$. Let $i : Rx \to R$ be the natural injection. Since Rx is finitely generated, R is projective and C is absolutely pure by Meggiben (1970) there exists $g : R \to C$ such that $gi = f$. Now $f \neq 0$ implies $g \neq 0$. Let $g' = g \mid T(R)$. Then $g' \neq 0$. But this is a contradiction because $T(R) \in \mathcal{T}$ and $\mathcal{C} \in \mathcal{F}$. Hence for every $x \in T(R)$, $Rx \in \mathcal{T}$.

Remark 2.7 : The converse of the above proposition is not necessarily true. Let $(\mathcal{T}, \mathcal{F})$ be the torsion theory of Example 2. 3. $(\mathcal{T}, \mathcal{F})$ is pseudo-hereditary but not cogenerated by any class of absolutely pure modules because over Z every absolutely pure module is injective.

Corollary 2.8—Every torsion theory over a regular ring is pseudo-hereditary.

PROOF : Since every torsion theory is cogenerated by its torsion free class and over a regular ring every module is absolutely pure, the corollary follows trivially from the above proposition.

Remark 2.9 : It is possible to give a direct proof of the above corollary.

Remark 2.10 : The converse of corollary 2.8 is not true. Let Z_4 be the ring of integers modulo 4. Then the only torsion theories of Z_4-mod are ($\{0\}$, Z_4-mod) and (Z_4-mod, $\{0\}$). Both of them are trivially pseudo-hereditary but Z_4 is not a regular ring.

We now give some applications of pseudo-hereditary torsion theory in the study of other types of torsion theories. Our first result gives an alternative proof on a result of Ramamurthi (1973).

Proposition 2.11—A cohereditary torsion theory $(\mathcal{T}, \mathcal{F})$ of R-mod is hereditary if and only if $R/T(R)$ is flat as a right R-module.

PROOF : It is enough to show that $R/T(R)$ is flat as a right R-module if and only if $(\mathcal{T}, \mathcal{F})$ is pseudo-hereditary. Now $(\mathcal{T}, \mathcal{F})$ is pseudo-hereditary if and only if $T(I) = T(R) \cap I$ for every left ideal I of R. Since $(\mathcal{T}, \mathcal{F})$ is already cohereditary by Beacly (1971), $T(I) = T(R).I$. Hence $(\mathcal{T}, \mathcal{F})$ is pseudo-hereditary if and only if $T(R).I = T(R) \cap I$ for every left ideal I of R. But by Lambek (1966), $T(R).I = T(R) \cap I$ for all left ideals I of R if and only if $R/T(R)$ is flat as a right R-module.

Corollary 2.12—If R is a fully idempotent ring then every cohereditary torsion theory is hereditary.

PROOF : The result follows from the fact that a ring R is fully idempotent if R/I is flat as a right R-module for every two sided ideal I of R (Fisher 1974).

In particular we have that every cohereditary torsion theory over a V-ring or a regular ring is hereditary.

Corollary 2.13—If R is a fully idempotent ring, then every *TTF* theory of R-mod is stable.

PROOF : Let $(\mathcal{C}, \mathcal{T})$ be the torsion theory associated to the *TTF* theory $(\mathcal{T}, \mathcal{F})$. Since $(\mathcal{C}, \mathcal{T})$ is cohereditary and R is fully idempotent by Corollary 2.12, $(\mathcal{C}, \mathcal{T})$ is hereditary and so $\mathcal{T}$ is closed under taking injective hulls. Hence $(\mathcal{T}, \mathcal{F})$ is stable.

Corollary 2.14—If R is a fully idempotent ring and $(\mathcal{T}, \mathcal{F})$ is a cohereditary torsion theory of R-mod, then $(\mathcal{T}, \mathcal{F})$ is centrally splitting if and only if $\mathcal{T}$ is closed under taking products.

PROOF : By corollary 2.12, $(\mathcal{T}, \mathcal{F})$ is a *TTF* theory. By Corollary 2.13, $(\mathcal{T}, \mathcal{F})$ is stable. The result now follows from the fact that a stable torsion theory which is both hereditary and cohereditary is centrally splitting.

3. Pseudo-hereditary Torsion Theories and Trivial Torsion Theories

For any ring R, the torsion theories (R-mod, $\{0\}$) and ($\{0\}$, R-mod) are trivially pseudo-hereditary. We now investigate the condition on a ring R so that these are the only pseudo-hereditary torsion theories of R-mod.

We first prove two simple results.

Lemma 3.1—Let $(\mathcal{T}, \mathcal{F})$ be the torsion theory cogenerated by the class of all simple R-modules. Let $(\mathcal{T}', \mathcal{F}')$ be a hereditary torsion theory such that $\mathcal{T}' \subseteq \mathcal{T}$. Then $\mathcal{T}' = \{0\}$.

Proof: Suppose $0 \neq M \in \mathcal{T}'$. So there exists $0 \neq x \in M$ such that $Rx \in \mathcal{T}'$. Since Rx is finitely generated, Rx has a maximal submodule N. Now $Rx/N \cong S$ for some simple R-module S and hence $Rx/N \in \mathcal{F}$. This implies $Rx = N$, which is a contradiction to the fact that N is a proper submodule of Rx. Hence $\mathcal{T}' = \{0\}$.

Lemma 3.2—Let R be a ring satisfying the following three conditions :

(1) R is self injective

(2) Every non-zero R-module contains a simple submodule.

(3) All simple R-modules are isomorphic.

Then the only faithful torsion theory of R-mod is the trivial torsion theory ($\{0\}$, R-mod).

Proof : Let $(\mathcal{T}, \mathcal{F})$ be a faithful torsion theory of R-mod. Then $R \in \mathcal{F}$. Since $R \neq 0$, R contains a simple submodule S. Since all simple R-modules are isomorphic, the torsion class $\mathcal{T}'$ of the torsion theory $(\mathcal{T}', \mathcal{F}')$ cogenerated by the class of all simple modules contains the torsion class $\mathcal{T}$ of $(\mathcal{T}, \mathcal{F})$. Now let $(\mathcal{T}'' \mathcal{F}'')$ be the torsion theory cogenerated by R. Since R is self injective, $(\mathcal{T}'', \mathcal{F}'')$ is hereditary. Moreover, $\mathcal{T} \subseteq \mathcal{T}'' \subseteq \mathcal{T}'$. Hence by the above lemma $\mathcal{T}'' = \{0\} = \mathcal{T}$.

We now characterize the class of rings over which the only pseudo-hereditary torsion theories are the trivial ones.

Proposition 3.3—The following are equivalent for a ring R.

(1) The only pseudo-hereditary torsion theories of R-mod are ($\{0\}$, R-mod) and (R-mod, $\{0\}$).

(2) (a) R is self injective.

(b) Every non-zero R-module contains a simple submodule.

(c) All simple R-modules are isomorphic.

Proof : (1) $\Rightarrow$ (2). Since the only pseudo-hereditary torsion theories are trivial ones, the only hereditary torsion theories are also the trivial ones. Hence by Gardner(1971), R satisfies the conditions (b) and (c) of (2). Now if $(\mathcal{T}, \mathcal{F})$ is a faithful torsion theory then, since $(\mathcal{T}, \mathcal{F})$ is triviallly pseudo-hereditary and $R \in \mathcal{F}$, $(\mathcal{T}, \mathcal{F})$ must be the torsion theory ($\{0\}$, R-mod). This implies that R is a cogenarator.

Again by Theorems 4 and 6 of Gardner (1971), $R/J(R)$ is semisimple artinian, where $J(R)$ is the Jacobson radical of R. Hence by Corollary 4.2, of Faith and Walker (1975). R is self injective.

(2) $\Rightarrow$ (1). Let $(\mathcal{T}, \mathcal{F})$ be a pseudo-hereditary torsion theory of R-mod. Then either $(\mathcal{T}, \mathcal{F})$ is faithful or it is non faithful. First suppose that $(\mathcal{T}, \mathcal{F})$ is non-faithful. Then $T(R) \neq 0$. So by condition 2 (b), there exists a simple R-module $S \subseteq T(R)$. Then $S \in \mathcal{T}$. Let $M \neq 0$ be any R-module. Then because of condition 2 (c), Hom $(S,M) \neq 0$. Hence $M \neq 0$ implies $M \notin \mathcal{F}$. So $\mathcal{F} = \{0\}$ and $\mathcal{T} = R$-mod.

If $(\mathcal{T}, \mathcal{F})$ is faithful then by Lemma 3.2, it is the trivial torsion theory ($\{0\}$ R-mod).

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