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A GENERALIZATION OF A FIXED POINT THEOREM OF JAGGI

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A result of Jaggi is generalized for a common fixed point of a pair of self-mappings on a metric space. The result is analogous to the Maiti and Pal's generalizations of a fixed point theorem of Ciric.

Adopting a clue from a generalization of Pal and Maiti (1977) for Ciric's fixed point theorem we give a generalization of a result of Jaggi (1977, Theorem 4) for common fixed point of a pair of maps T_1 and T_2.

Theorem 1 — Let T_1 and T_2 be two self maps defined on a complete metric space (X, d) satisfying any one of the following conditions :

$$\text{(a)} \quad \frac{d(x, T_1x)\, d(y, T_2y)}{d(T_1x, T_2y)} + d(T_1x, T_2y) \leqslant \alpha\, d(x, y),\ 1 \leqslant \alpha < 2$$

$$\text{(b)} \quad \frac{d(x, T_1x)\, d(y, T_2y)}{d(T_1x, T_2y)} + d(T_1x, T_2y) \leqslant \beta \{d(x, T_2y) + d(y, T_1x) + d(x, y)\},\ \tfrac{1}{2} \leqslant \beta < \tfrac{2}{3}$$

with $T_1x \neq T_2y\ \forall\ x, y \in X$.

Then T_1, T_2 have a common fixed point.

PROOF : Starting from an arbitrary point $x_0 \in X$ we define sequence $\{x_n\}$ by

$$\left.\begin{aligned} x_{2n+1} &= T_1 x_{2n} \\ x_{2n+2} &= T_2 x_{2n+1} \end{aligned}\right\} n = 0, 1, 2, \ldots$$

Then in case T_1, T_2 satisfy (a) we have

$$\frac{d(x_0, T_1 x_0)\, d(x_1, T_2 x_1)}{d(T_1 x_0, T_2 x_1)} + d(T_1 x_0, T_2 x_1) \leqslant \alpha\, d(x_0, x_1)$$

or

$$d(x_1, x_2) \leqslant (\alpha - 1)\, d(x_0, x_1)$$

In general,

$$\frac{d(x_{2n}, T_1 x_{2n})\, d(x_{2n+1}, T_2 x_{2n+1})}{d(T_1 x_{2n}, T_2 x_{2n+1})} + d(T_1 x_{2n}, T_2 x_{2n+1}) \leqslant \alpha\, d(x_{2n}, x_{2n+1})$$

or

$$d(x_{2n+1}, x_{2n+2}) \leqslant (\alpha - 1)\, d(x_{2n}, x_{2n+1})$$
$$\leqslant (\alpha - 1)^{2n+1}\, d(x_0, x_1) \qquad \ldots(1)$$

Similarly

$$d(x_{2n}, x_{2n+1}) \leqslant (\alpha - 1)^{2n}\, d(x_0, x_1). \qquad \ldots(2)$$

Since $(\alpha - 1) < 1$, we find that $\{x_n\}$ is a Cauchy sequence and therefore there is a point $u \in X$ such that $x_n \to u$ as $n \to \infty$. Also

$$T_1 x_{2n} = x_{2n+1} \to u \text{ and}$$

$$T_2 x_{2n+1} = x_{2n+2} \to u.$$

Now putting $x = u$ and $y = x_{2n+1}$ in (a) we have

$$\frac{d(u, T_1 u)\, d(x_{2n+1}, T_2 x_{2n+1})}{d(T_1 u, T_2 x_{2n+1})} + d(T_1 u, T_2 x_{2n+1}) \leqslant \alpha\, d(u, x_{2n+1}).$$

Taking limit we get

$$\frac{d(u, T_1 u)\, d(u, u)}{d(T_1 u, u)} + d(T_1 u, u) \leqslant \alpha\, d(u, u)$$

or

$$d(T_1 u, u) \leqslant 0.$$

This implies that $T_1 u = u$ i.e. u is a fixed point of T_1. Similarly putting $x = x_{2n}$ and $y = u$ in (a) we can prove that $T_2 u = u$. Hence T_1, T_2 have a common fixed point.

In the same fashion if T_1, T_2 satisfy (b) then the inequalities (1) and (2) respectively become

$$d(x_{2n+1}, x_{2n+2}) \leqslant \left(\frac{2\beta - 1}{1 - \beta}\right)^{2n+1} d(x_0, x_1) \qquad \ldots(3)$$

$$d(x_{2n}, x_{2n+1}) \leqslant \left(\frac{2\beta - 1}{1 - \beta}\right)^{2n} d(x_0, x_1) \qquad \ldots(4)$$

and the arguments as given above leads to the existence of a common fixed point u of T_1 and T_2 in this case also.

Corollary — Let T be a self map defined on a complete metric space (X, d). Further, let T satisfy either of the following conditions :

(i) $$\frac{d(x, Tx)\, d(y, Ty)}{d(Tx, Ty)} + d(Tx, Ty) \leqslant \alpha\, d(x, y),\ 1 \leqslant \alpha < 2$$

(ii) $$\frac{d(x, Tx)\, d(y, Ty)}{d(Tx, Ty)} + d(Tx, Ty) \leqslant \beta \{d(x, Ty) + d(y, Tx) + d(x, y)\},\ \tfrac{1}{2} \leqslant \beta < \tfrac{2}{3}$$

with $Tx \neq Ty$

Then T has a fixed point in X.

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