Indian J. pure appl. Math., **14**(5) : 596-603, May 1983

ON SPECTRA AND NUMERICAL RANGES IN LOCALLY m-CONVEX ALGEBRAS

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(*Received* 16 *June* 1982)

The consequences of the assumptions of boundedness and compactness of the spectrum and the numerical range of every element of a complete locally multiplicatively convex algebra are discussed. These give certain characterizations of Q-algebras, Banachable algebras and Banachizable algebras among complete locally multiplicatively convex algebras.

A linear associative algebra A (throughout assumed to be with identity 1) is called a 'locally multiplicatively convex (lmc) topological algebra' if it carries a topology t which is determined by a family $P = (p_\alpha \mid \alpha \in \Delta)$ of seminorms satisfying $p_\alpha(xy) \leqslant p_\alpha(x)\, p_\alpha(y)$ for all x, y in A. We can, without loss of generality, assume that $p_\alpha(1) = 1$ for all α. Such a family P is called a 'unital m-calibration' on A. Such an algebra has been introduced by Michael (1952), who called it a 'Q-algebra' if additionally the set A^{-1} of invertible elements of A is an open set. For a commutative lmc algebra (A, t), let χ_A (respectively Σ_A) denote the set of all non-zero multiplicative linear functionals (respectively, all non-zero continuous multiplicatively linear functionals) on A with the usual topology induced by the weak topology $\sigma(A^*, A)$ on A^*, the algebraic dual of A. For $a \in A$, let $sp(a)$ be the spectrum of a in A $sp(a) = \{\lambda \in \mathbb{C} \mid a - \lambda 1$ is not invertibe in $A\}$, and let $r(a) = \sup\{\mid \lambda \mid \lambda \in sp(a)\}$.

Recently Zelazko (1976) has discussed the consequences of the boundedness of $sp(a)$ for each $a \in A$ in a commutative lmc algebra A. The following theorem extends his results to the non-commutative case.

Theorem 1—Let A be a complete lmc algebra. The following are equivalent.

(a) For each $a \in A$, $sp(a)$ is bounded.

(b) For each $a \in A$, $sp(a)$ is compact.

(c) A is a complete lmc Q-algebra under some topology finer than that of A,

(d) A is a complete lmc Q-algebra under some topology.

If further A is commutative, then any of above is equivalent to :

(e) χ_A is compact.

(f) Every maximal ideal of A is of codimension one.

PROOF : We need to modify Zelazko's arguments. Let $a \in A$. Let M_a be a maximal abelian subalgebra containing a. Exactly as in Bonsall and Duncan (1973, Proposition 15.3), M_a is closed, hence complete and $sp_A(a) = sp_{M_a}(a)$. Now Theorem 1 in Zelazko (1976) gives $(a) \Leftrightarrow (b)$. To show that $(b) \Rightarrow (c)$, let $U = \{x \in A \mid r(x) < 1\}$, $V = \overline{|co|}\ (\bigcup_{n=1}^{\infty} U^n)$, $\overline{|co|}$ denoting the closed absolutely convex hull. By (b), U and so V is absorbing which is clearly absolutely convex and idempotent. Hence $p_V(x) = \inf\{\lambda > 0 \mid x \in \lambda V\}$ defines a submultiplicative seminorm on A. If P is any m-calibration for the topology t of A, then clearly $P' = P \cup \{p_V\}$ defines a finer lmc topology t_1 on A. Since t_1 has a neighbourhood base consisting of sets which are complete in t, it follows from Schaeffer (1966, Chap I, Theorem 1.6) that (A, t_1) is complete. Also, since U is a t_1-neighbourhood of 0, (A, t_1) is a Q-algebra. Thus $(b) \Rightarrow (c) \Rightarrow (d)$. Assume (d). Let $a \in A$. Due to the maximality of M_a, $M_a^{-1} = M_a \cap A^{-1}$. Thus M_a with the relative t_1-topology is a Q-algebra, and so (a) follows. The final part of the theorem is a part of Zelazko's theorem.

Corollary 2—Let A be a barrelled complete lmc algebra. The following are equivalent.

(a) For each $a \in A$, $sp(a)$ is bounded.

(b) A is a Q-algebra.

If A is also commutative, then either of above is equivalent to :

(c) Σ_A is compact.

This follows from Michael (1952, Theorem 13.6) See also Zelazko (1976, Corollary 3, p. 296).

Now we aim to discuss the numerical range analogue of Theorem 1. Let (A, t) be a complete lmc algebra. Let $P \in \mathscr{P}_{um}(A)$, the unital m-calibrations determining the topology t, say $P = (p_\alpha \mid \alpha \in \Delta)$ be directed. Let A' be the dual of (A, t). For each α, let

$$D(A, p_\alpha, 1) = \{f \in A' \mid f(1) = 1, \mid, f(x) \mid \leqslant p_\alpha(x) \text{ for all } x\}.$$

Let $D(A, P, 1) = \bigcup_\alpha D(A, p_\alpha, 1)$, the 'state space' (or better P-state space) of A. The numerical range of $a \in A$ w. r. t. P—the 'P-numerical range'—is $V(A, P, a) = \{f(a) \mid f \in D(A, P, 1)\}$.

This definition is due to Giles and Koehler (1971). They have shown that

$$B_P = \{a \in A \mid V(A, P, a) \text{ is bounded}\}$$
$$= \{a \in A \mid \sup_\alpha p_\alpha(a) < \infty\}$$

is a subalgebra of A and $p(a) = \sup_\alpha p_\alpha(a)$ defines an algebra norm on B_P. If A is complete, then (B_P, p) is a Banach algebra. Also, for $a \in B_P$, $\overline{V(A, P, a)} = V(B_P, p, a)$ where $V(B_P, p, a)$ is the Banach algebra numerical range of a in (B_P, p). For numerical ranges in Banach algebras, we refer to Bonsall and Duncan (1971 and 1973).

Theorem 3—Given an (A, P), the following are equivalent.

(a) For each $a \in A$, $V(A, P, a)$ is bounded.

(b) The set $D(A, P, 1)$ is relatively $\sigma(A^*, A)$ compact.

In either case, there is a Banach algebra topology on A finer than t. Conversely, if A admits Banach algebra topology finer than t, then there exists a P in $\mathscr{P}_{um}(A)$ such that $V(A, P, a)$ is bounded for all a in A.

PROOF : Assume (a). Then $A = B_P$ and the topology due to p is clearly finer than the given lmc topology t of A. Since the set $D(A, p, 1) = \{f \in (A, p)' \mid f(1) = 1, |f(x)| \leqslant p(x)$ for all $x \in A\}$ is known to be weak* compact by Bonsall and Duncan (1971, Theorem 2.3), it suffices to prove that $\overline{D(A, P, 1)} = D(A, p, 1)$ where the bar denotes the $\sigma(A^*, A)$ closure in $(A, p)'$. Obviously, $\overline{D(A, P, 1)} \subset D(A, p, 1)$. If there exists an f_0 in $D(A, p, 1) \setminus \overline{D(A, P, 1)}$, then by the standard Hahn-Banach argument, there is an $a \in A$ and nonzero reals δ_1 and δ_2 such that $\operatorname{Ref}(a) < \delta_1 < \delta_2 < \operatorname{Re} f_0(a)$ for all $f \in D(A, P, 1)$. Taking supremum over f, the numerical radious formula (Giles and Koehler 1973, Lemma 2) and its Banach algebra counter part (Bonsall and Duncan 1971, Theorem 2.5) give

$$\inf_{\beta>0} \frac{1}{\beta}\left\{\sup_\alpha p_\alpha(1 + \beta a) - 1\right\} < \inf_{\beta>0} \frac{1}{\beta}\left\{p(1 + \beta a) - 1\right\}$$

which is a contradiction. Hence $\overline{D(A, P, 1)} = D(A, p, 1)$. This with the other obvious implication give $(a) \Leftrightarrow (b)$.

Now suppose there exists a Banach algebra norm $\|\cdot\|$ on A such that the identity map $id : (A, \|\cdot\|) \to (A, t)$ is continuous. Then $S = \{a \in A \mid \|a\| \leqslant 1\}$ is a t-bounded idempotent in A. By a recalibration lemma (Joseph 1977, Lemma 2.3), there exists a unital m-calibration $P = (p_\alpha)$ for (A, t) such that $S \subset S_P = \{a \in A \mid p(a) = \sup_\alpha p_\alpha(a) \leqslant 1\}$. (If necessary, we can take the directed hull of P viz, add to P the maxima of every of its finite subfamily). Since S is absorbing, so is S_P and $A = B_P$. It follows that $V(A, P, a)$ is bounded for all $a \in A$. This completes the proof.

The general open mapping theorem (Schaeffer, 1966, Chap. I, § 8) immediately gives the following analogue of Corollary 2.

Corollary 4—Let (A, t) be a complete barrelled lmc algebra. If there exists an m-calibration P on A such that $V(A, P, a)$ is bounded for all a in A, then the topology t of A is normable.

Next we consider the compactness of the numerical ranges. The following is suggested by Theorem 1.

Question 5—Given an (A, P), is it true that $V(A, P, a)$ is bounded for all $a \in A$ iff $V(A, P, a)$ is compact for all $a \in A$?

We affirmatively answer this for a particular class of A. Recall (Michael 1952, § 6) that a 'b^*-algebra' A with a b^*-calibration $P = (p_\alpha)$ is a calibrated complete lmc algebra (A, P) with an involution $x \to x^*$ such that $p_\alpha(x^* x) = p_\alpha(x)^2$ for all α and for all x.

Theorem 6—Let A be a commutative b^*-algebra with a b^*-calibration $P = (p_\alpha)$. If $sp(a)$ is bounded for all a in A (in particular, if $V(A, P, a)$ is bounded for all a in A), then $V(A, P, a)$ is compact for all a in A.

For the proof, we shall need lmc analogue of Vidav's lemma (Bonsall and Duncan 1971, Theorem 5. 14). Given a complete lmc algebra A with an m-calibration $P = (p_\alpha)$, let, for each α, $N_\alpha = \{x \in A \mid p_\alpha(x) = 0\}$, a two sided ideal of A. Let $A_\alpha = A/N_\alpha$, a normed algebra with norm $\tilde{p}_\alpha(x_\alpha) = p_\alpha(x)$ for all $x_\alpha = x + N_\alpha$ in A_α. Let $\tilde{A}_\alpha$ be its completion. Let $\pi_\alpha : A \to A_\alpha$ be the quotient map. Then as in Michael (1952), $(\tilde{A}_\alpha, \tilde{p}_\alpha)$ forms a projective system of Banach algebras and $(A, t) = \varprojlim \tilde{A}_\alpha$, projective limit of A_α' s. The set $D(A, p_\alpha, 1)$ is canonically isomorphic with the state space $D(\tilde{A}_\alpha, \tilde{p}_\alpha, 1_\alpha)$ of $\tilde{A}_\alpha$ under the correspondence $f \to \tilde{f}_\alpha$ where $\tilde{f}_\alpha$ on A_α is defined by $\tilde{f}_\alpha(x_\alpha) = f(x)$ $(x \in A)$ and extended continuously to $\tilde{A}_\alpha$. This gives the Banach algebra numerical range of a_α in $\tilde{A}_\alpha$ as $V(\tilde{A}_\alpha, \tilde{p}_\alpha, a_\alpha) = \{f(a) \mid f \in D(A, p_\alpha, 1)\}$. In analogy with Bonsall and Duncan (1971, Definition 5.13), we call an $a \in A$ 'P-normal' if $a = h + ik$, $hk = kh$ where h and k are in $H(A, P) = \{x \in A \mid V(A, P, x) \subset \mathbf{R}\}$, the P-hermitian elements of A.

Lemma 7—Let A be a complete lmc algebra with an m-calibration P. For every P-normal element x of A, $V(A, P, x) = \cos p(x)$, the convex hull of $sp(x)$.

PROOF : Let x be P-normal with $x = h + ik$; h, k in $H(A, P)$. Since $V(A, P, x) = \bigcup_\alpha V(\tilde{A}_\alpha, \tilde{p}_\alpha, a_\alpha)$, it is easily seen that for each α, h_α and k_α are hermitian elements of the Banach algebra $(\tilde{A}_\alpha, \tilde{p}_\alpha)$, $x_\alpha = h_\alpha + ik_\alpha$ and $h_\alpha k_\alpha = k_\alpha h_\alpha$. Thus x_α being normal in $\tilde{A}_\alpha$, Vidav's lemma (Bonsall and Duncan 1971, Theorem 5.14) implies that for each α, $(V_{\tilde{A}_\alpha}(\tilde{p}_\alpha, x_\alpha) = \cos p_{\tilde{A}_\alpha}(x_\alpha)$. Then

$$V(A, P, x) = \bigcup_{\alpha} V(\tilde{A}_\alpha, \tilde{p}_\alpha, x_\alpha)$$

$$= \bigcup_{\alpha} \cos p_{\underset{A_\alpha}{\sim}}(x_\alpha)$$

$$\subset \cos\left(\bigcup_{A_\alpha} sp_{\underset{A_\alpha}{\sim}}(x_\alpha)\right)$$

$$= \cos p_A(x) \text{ by Michael (1952, Corollary 5.3)}$$

$$\subset V(A, P, x) \text{ by Giles and Koehler (1973, Theorem 1).}$$

Hence the lemma follows.

PROOF OF THEOREM 6 : Since P is a b^*-calibration on A, by Giles and Koehler (1973, Theorem 6), the involution $x \to x^*$ of A is determined by the P-hermitian decomposition $A = H(A, P) + iH(A,P)$. Hence for any $x \in A$, x is normal w. r. t. the involution ($xx^* = x^*x$) iff x is P-normal. By Theorem 1 $(a) \Leftarrow\Rightarrow (b)$ and Lemma 7, $V(A, P, x)$ is compact for all normal x. Since A is commutative, the result follows.

The following example shows that Theorem 6 does not hold for a complete lmc algebra that is not a b^*-algebra.

Example 8—Let $A = C^\infty[a, b]$, the algebra with pointwise operations of all infinitely differentiable complex valued functions on a closed and bounded internal $[a, b]$ with the complete lmc topology determined by the calibration $P_n = (p \mid n = 0, 1, 2, 3, \ldots)$ where

$$p_n(f) = \sup\left\{\sum_{k=0}^{n} \frac{|f^{(k)}(t)|}{k!} \mid a \leqslant t \leqslant b\right\}.$$

For each $f \in C^\infty[a, b]$, $sp(f)$ = range of f which is bounded. By Bonsall and Duncan (1973, Corollary 18.22), there is no norm on $C^\infty[a, b]$ that makes it a Banach algebra. Hence in view of Theorem 3 it is not true that $V(A, P, f)$ is bounded for all $f \in C^\infty[a, b]$.

The following example shows that even if A is a b^*-algebra and $V(A, P, a)$ is compact for all $a \in A$, A need not be Banachizable.

Example 9—Let X be a pseudo-compact, non-compact, completely regular Hausdorff space which is locally compact or first countable. Let A be $(C(X), k)$, the complete lmc algebra of all continuous complex valued functions on X with the compact-open topology k. The topology k is determined by the m-calibration $P = \{p_K \mid K \subset X \text{ compact}\}$ where $p_K(f) = \sup\{|f(x)| \mid x \in K\}$.

Since each $f \in C(X)$ is bounded, in our notation, $p(f) = \sup\{|f(t)| : t \in X\}$ which defines a Banach algebra topology on A strictly finer than k; and so k is not normable.

Infact, for each $f \in C(X)$, $sp(f)$ = range of f which by Theorem 1 $(a) \Leftarrow\Rightarrow (b)$ is compact. Since A is a b^*-algebra with b^*-calibration P, Theorem (6) implies that

$V(A, P, f)$ is compact for all $f \in C(X)$. By Giles and Koehler (1973, Theorem 3), $V(A, P, f) = V(A, p, f)$.

Further, as is well known,

$D(A, P, 1)$ = set of all k-continuous positive linear functionals on A.

$D(A, p, 1)$ = set of all positive linear functionals on A,

and $D(A, P, 1) \neq D(A, p, 1)$. By Brooks (1966, Theorem 4.4), $D(A, P, 1)$ is closed in $(A, P)'$. However, it is not closed in $(A, p)'$. As in Theorem 3, $\overline{D(A, P, 1)} = D(A, p, 1)$ which shows that the compactness of $V(A, P, f)$ for all f does not imply the weak compactness of $D(A, P, 1)$.

Infact, by Brooks (1966, Theorem 5.1),

$(A, P)'$ = all complex regular Borel measures on X with compact support,

whereas, as is well known,

$(A, p)'$ = all complex regular Borel measures on the Stone-Cech compactification βX of X.

This shows that the topologies on A due to P and p are not even consistent with the duality in the sense of Schaeffer (1966, Chap. IV, § 3).

In this direction we present the following.

Proposition 10—Let A be a complete lmc algebra with an m-calibration P. If $D(A, P, 1)$ is $\sigma(A^*, A)$ compact, then $V(A, P, a)$ is compact for all a in A and $(A, P)' = (A, p)'$.

PROOF: If $D(A, P, 1)$ is $\sigma(A^*, A)$ compact, then by the continuity of $x \to f(x)$, $V(A, P, x)$ is compact for all $x \in A$. As in Theorem 3, $A = B_P$ and $D(A, p, 1) = \overline{D(A, P, 1)} = D(A, P, 1)$. The lemma that follows together with its Banach algebra counterpart (Bonsall and Duncan 1973, Lemma 31.1) gives the desired conclusion.

Lemma 11—Let A be a complete lmc algebra with an m-calibration P. Then the dual A' of A is a complex linear span of $D(A, P, 1)$.

We omit the straight forward proof of the lemma which requires the standard lmc technique of passing to the factor algebras $\tilde{A}_\alpha$ as discused prior to Lemma 7 and of applying Bonsall and Duncan (1973, Lemma 31.1)

Question 12—Does the converse of Proposition 10 hold ?

The following theorem characterizes Banach algebras among complete lmc algebras A. In view of the fact (Schaeffer 1966, Chap. III, Corollary to Theorem 4.3) that an equicontinuous subset of A' is relatively $\sigma(A', A)$ compact but not conversely, the theorem that follows implies that even the compactness of $D(A, P, 1)$ need not suffice for the Banachizability A. For $a \in A$, $v(a) = \sup\{|\lambda| \mid \lambda \in V(A, P, a)\}$ is the numerical radius of a in A.

Theorem 13—Let (A, t) be a complete lmc algebra with an m-calibration P. The following are equivalent.

(a) $a \to \overline{V(A, P, a)}$ is set-valued upper semi-continuous on A.

(b) $a \to V(A, P, a)$ is set-valued upper semi-continuous on A.

(c) $a \to v(a)$ is an upper semi-continuous real valued function on A.

(d) $a \to v(a)$ is a continuous real valued function on A.

(e) $D(A, P, 1)$ is equicontinuous.

(f) The topology t of A is normable and is determined by p.

PROOF : Assume (a). Let $x_0 \in A$. Let V_0 be a neighbourhood of $V(A, P, x_0)$. Choose a neighbourhood W_0 of $V(A, P, x_0)$ such that $V(A, P, x_0) \subset W_0 \subset \overline{W}_0 \subset V_0$. Then $\overline{V(A, P, x_0)} \subset V_0$. By (a), there is a neighbourhood U of x_0 in (A, t) such that $\overline{V(A, P, x)} \subset V_0$ for all $x \in U$. This immediately gives (b).

Assume (b). Since $V(A, P, 0) = \{0\}$, there is a neighbourhood U of 0 in A such that $V(A, P, a) \subset \{z \in \mathbb{C} \mid |z| < 1\}$ for all $a \in U$. As U is absorbing, and $V(A, P, \lambda a) = \lambda V(A, P, a)$, $V(A, P, a)$ is bounded for all $a \in A$. Hence $v(a) < \infty$ for all $a \in A$. Again by (b), $\{a \in A \mid v(a) < k\}$ (k any constant) is a neighbourhood of 0 in (A, t). This gives (c) from which (d) is immediate, as, in view of (Giles and Koehler 1973, Lemma 1), $a \to v(a)$ is always lower semi-continuous. Further $|f(a)| \leqslant v(a)$ for all $a \in A, f \in D(A, P, 1)$ shows that $(d) \Rightarrow (e)$.

Assume (e). By Schaeffer (1966, Chap. III, Theorem 4.1), $D(A, P, 1)$ is $\sigma(A^*, A)$ bounded, and so $V(A, P, a)$ is bounded for all $a \in A$. As in Theorem 3, $A = B_P$, $p(a) < \infty$ $(a \in A)$ and $\overline{D(A, P, 1)} = D(A, p, 1)$. Since the pointwise closure of an equi-continuous set of linear map is equi-continuous (Schaeffer 1966, Chap. III, Theorem 4.5), it follows that $D(A, p, 1) \subset (A,P)'$ — it is, infact, equicontinuous. Hence $\{a \in A \mid v(a) < 1\}$ is a neighbourhood of 0 in (A, P). Since $\overline{V(A, P, a)} = V(A, p, a)$, $v(a) = \sup\{|\lambda| \mid \lambda \in V(A, p, a)\}$. But $a \to v(a)$ is a linear norm on A equivalent to p by Bonsall and Duncan (1971, Theorem 4.1). It now follows that the topology t is determined by p.

That $(b) \Leftrightarrow (a)$ follows from Giles and Koehler (1973, Theorem 3) and the upper semi-continuity of the numerical range in Banach algebra (Bonsall and Duncan 1971, § 2). This completes the proof.

The following provides a spectral analogue of above theorem. We omit the proof which is along the same lines as above. Here $r(x) = \sup\{|\lambda| \mid \lambda \in sp(x)\}$ is the spectral radius of x in A.

Theorem 14—Let A be a complete lmc algebra. The following are equivalent.

(a) $a \to sp(a)$ is upper semicontinuous on A.

(b) $a \to r(a)$ is a continuous real valued function on A.

(c) A is a Q-algebra-

Further, if A is commutative, then any of the above is equivalent to :

(d) Σ_A is equicontinuous.

It is well known that if $(A, \| \cdot \|)$ is normed algebra, then for each $x \in A$, the limit $\lim_{n\to\infty} \| x^n \|^{1/n} = r'(x)$ (say) exists and $r'(x) = \inf \| x^n \|^{1/n}$. Also, $r'(x) \leqslant r(x)$; and if A is a Banach algebra, then $r'(x) = r(x)$ for all $x \in A$. The followiug shows that this spectral radius formula characterizes Q-algebras among normed algebras.

Proposition 15—A normed algebra $(A, \| \cdot \|)$ is a Q-algebra if and only if $r'(x) = r(x)$ holds for every x in A.

PROOF : The continuity of $x \to \| x \|$ on A gives the upper semi-continuity of $x \to r'(x)$. Hence if $r'(x) = r(x)$ holds, then $U = \{ x \in A \mid r(x) < 1 \}$ is a neighbourhood of 0 in $(A, \| \cdot \|)$, so by Michael (1952, Proposition 13.5), A is a Q-algebra,

Conversely, let A be a Q-algebra. Then above U is a neighbourhood of 0. Hence it is absorbing and $sp(x)$ is bounded for all $x \in A$. For the moment, assume A to be commutative. Then, since in a Q-algebra maximal ideals are closed, $\chi_A = \Sigma_A$. Hence given $f \in \chi_A$, there is a constant k such that $| f(x) | \leqslant k \| x \|$ for all $x \in A$. Then $| f(x) |^n = | f(x^n) | \leqslant k \| x^n \|$, and so $| f(x) | \leqslant r'(x)$ for all $f \in \chi_A$. But in a commutative Q-algebra, as is easy to see, $sp(x) = \{ f(x) \mid f \in \chi_A \}$ for any $x \in A$. Hence $r(x) \leqslant r'(x)$, $r(x) = r'(x)$ for all $x \in A$.

In the general case, given $x \in A$, if M_x is a maximal abelian subalgebra of A containing x, then $M_x^{-1} = A^{-1} \cap M_x$ implies that M_x is a Q-algebra. Also $sp_A(x) = sp_{M_x}(x)$. Then $r(x) = r_{M_x}(x) = r'(x)$ by above, as $r'(x)$ depends only the norms of powers of x. Hence the result follows.

ACKNOWLEDGEMENT

It is a pleasure to express sincere thanks to Dr M. H. Vasavada for discussion during the preparation of this paper. Thanks are also due to the referee for some suggestions.

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