Indian J. pure appl. Math., **14**(5) : 641-645, May 1983

CERTAIN TRIPLE SERIES EQUATIONS INVOLVING JACOBI POLYNOMIALS

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(*Received* 8 *April* 1982; *after revision* 16 *November* 1982)

Solution of a new class of triple series equations involving Jacobi polynomials has been obtained in the form of Fredholm integral equation. This is a slightly more general problem than that considered by Srivastava (1979, 1980).

1. INTRODUCTION

In the past two decades various workers have investigated the solution of triple series equations involving trigonometric series, the Fourier-Bessel series, the Fourier-Legendre series, and series of Jacobi and Laguerre polynomials. Some of these series arise in the solution of mixed boundary value problems of mathematical physics (Sneddon 1966). Recently, Srivastava (1979, 1980) obtained the solution of dual series equations involving Jacobi polynomials. Here we generalize his problem to discuss the solution of the following sets of triple series equations :

(i) Triple series equations of the first kind :

$$\sum_{n=0}^{\infty} A_n \frac{\Gamma(\lambda + n + l + 1)}{\Gamma(\gamma + n + l + 1)} P_{n+l}^{(\gamma, \delta)}(x) = 0, \quad -1 \leqslant x \leqslant y, \; z < x \leqslant 1 \qquad \ldots(1.1)$$

$$\sum_{n=0}^{\infty} A_n \frac{\Gamma(\mu + n + l + 1)}{\Gamma(\beta + n + l + 1)} P_{n+l}^{(\alpha, \beta)}(x) = f(x), \; y < x < z \qquad \ldots(1.2)$$

(ii) Triple series equations of the second kind :

$$\sum_{n=0}^{\infty} A_n \frac{\Gamma(\mu + n + l + 1)}{\Gamma(\beta + n + l + 1)} P_{n+l}^{(\alpha, \beta)}(x) = 0, \quad -1 \leqslant x < y, \; z < x \leqslant 1 \qquad \ldots(1.3)$$

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$$\sum_{n=0}^{\infty} A_n \frac{\Gamma(\lambda + n + l + 1)}{\Gamma(\gamma + n + l + 1)} P_{n+l}^{(\gamma, \delta)}(x) = g(x), \; y < x < z. \qquad ...(1.4)$$

Here l is an arbitrary non-negative integer; $f(x)$, $g(x)$ are prescribed functions and in general

$$\min(\alpha, \beta, \gamma, \delta, \lambda, \mu) > -1. \qquad ...(1.5)$$

The method employed is an interesting modification of the familiar technique developed by Lowndes (1968, 1969) for certain other sets of triple series equations.

The triple series equations (1.1) to (1.4) are also generalisations of those considered earlier by Collins (1962).

In the course of our analysis we shall use the following result which can easily be deduced:

$$M(x, \xi) = \sum_{n=0}^{\infty} \frac{\Gamma(\mu + n + l + 1)\,(n + l)!\,(\gamma + \delta + 2\eta + 2l + 1)}{\Gamma(\beta + n + l + 1)\,\Gamma(\lambda + n + l + 1)\,2^{\gamma+\sigma+1}}$$

$$\times \frac{\Gamma(\gamma + \quad + n + l + 1)}{\Gamma(\delta + n + l + 1)} P_{n+l}^{(\alpha, \beta)}(x) P_{n+l}^{(\gamma, \delta)}(\xi)$$

$$= a_n^* (1 + x)^{-\beta}(1 + \xi)^{-\delta} \int_{-1}^{w} \eta(t)\,(x - t)^{\gamma+\rho-\alpha}\,(\xi - t)^{\rho-1}\,dt$$

$$= a_n^* (1 + x)^{-\beta}(1 + \xi)^{-\delta} M_w(x, \xi), \; w = \min(x, \xi)$$

where

$$\eta(t) = (1 + t)^{\delta-\rho}(1 - t)^{-\gamma-\rho}$$

$$a_n^* = \frac{\Gamma(\mu + n + l + 1)\,(\gamma + \delta + 2n + 2l + 1)\,\Gamma(\gamma + \delta + n + l + 1)}{\Gamma(\lambda + n + l + 1)\,2^{\gamma+\delta+1}\,\Gamma(\delta - \rho + n + l + 1)\,\Gamma(\rho)}$$

$$\times \frac{2^{\alpha+\beta+1}\,\Gamma(\gamma + \rho + n + l + 1)}{\Gamma(\gamma + \rho - \alpha)\,(\alpha + \beta + 2n + 2l + 1)\,\Gamma(\alpha + \beta + n + l + 1)}$$

it being assumed that the parameters are so constrained that a_n^* is independent of n. This is, of course, possible when for instance, $\mu = \lambda$, $\alpha + \beta = \gamma + \delta$, $2\rho = \delta - \gamma$.

2. Solution of Triple Series Equations

First we shall consider the sequence $\{A_n\}$ satisfying the triple series equations of the first kind (1.1) and (1.2).

By employing the technique developed by Lowndes (1968, 1969) we obtain the solution of the above equations as below:

$$A_n = \frac{(n+l)!\,(\gamma+\delta+2n+2l+1)\,\Gamma(\gamma+\delta+n+l+1)}{\Gamma(\lambda+n+l+1)\,2^{\gamma+\delta+1}\,\Gamma(\delta+n+l+1)}$$

$$\times \int_y^z (1-\xi)^\gamma (1+\xi)^\delta P_{n+l}^{(\gamma,\delta)}(\xi)\,\varphi(\xi)\,d\xi,\ n = 1,2, \qquad \ldots(2.1)$$

where $\varphi(\xi)$ is given in terms of $\bar{\varphi}(\xi)$ as

$$\varphi(\xi)\,(1-\xi)^\gamma = -\frac{\sin(1-\rho)\pi}{\pi}\frac{d}{d\xi}\int_\xi^z \frac{\bar{\varphi}(t)\,dt}{(t-\xi)^\rho},\ y < \xi < z \qquad \ldots(2.2)$$

and $\bar{\varphi}(t)$ is determined from the Fredholm integral equation

$$\eta(t)\,\bar{\varphi}(t) = F(t) - \frac{\sin(1+\alpha-\gamma-\rho)\pi}{\pi}$$

$$\times \int_{-1}^{y} \frac{(y-r)^{\gamma+\rho-\alpha}\,\eta(\gamma)\,\bar{\varphi}(\gamma)\,d\gamma}{(t-r)\,(t-y)^{\gamma+\rho-\alpha}} \qquad \ldots(2.3)$$

where $\eta(t)$ is defined as earlier and

$$F(t) = \frac{\sin(1+\alpha-\gamma-\rho)\pi}{\pi\,a_n^*}\frac{d}{dt}\int_y^t \frac{K}{(1+x)^\beta}\frac{f(x)\,dx}{(t-x)^{\gamma+\rho-\alpha}}. \qquad \ldots(2.4)$$

The above analysis is formally true under the conditions (1.5) and the additional conditions

$$\mu = \lambda,\ \alpha+\beta = \gamma+\delta,\ 2\rho = \delta-\gamma. \qquad \ldots(2.5)$$

Next, we consider the triple series equations of the second kind (1.3) and (1.4). On proceeding as earlier, we get the solution of these equations as

$$B_n = \frac{(n+l)!\,(\gamma+\delta+2n+2l+1)\,\Gamma(\gamma+\delta+n+l+1)}{2^{\gamma+\delta+1}\,\Gamma(\lambda+n+l+1)\,\Gamma(\delta+n+l+1)}$$

$$\times\Big[\int_{-1}^{y}(1-\xi)^\gamma(1+\xi)^\delta\,\psi_1(\xi)P_{n+l}^{(\gamma,\delta)}(\xi)\,d\xi$$

$$+\int_z^1 (1-\xi)\,(1+\xi)^\delta\psi_2(\xi)P_{n+l}^{(\gamma,\delta)}(\xi)\,d\xi\Big] \qquad \ldots(2.6)$$

where ψ_1, ψ_2 are determined from the following sets of equations:

$$\psi_1(\xi)(1-\xi)^\gamma = -\frac{\sin(1-\rho)\pi}{\pi}\frac{d}{d\xi}\int_\xi^y \frac{\bar{\psi}_1(t)\,dt}{(t-\xi)^\rho},\ -1 < \xi < y \qquad \ldots(2.7)$$

$$\psi_2(\xi)(1-\xi)^{\gamma} = -\frac{\sin(1-\rho)\pi}{\pi}\frac{d}{d\xi}\int_{\xi}^{1}\frac{\bar{\psi}_2(t)\,dt}{(t-\xi)^{\rho}}, \quad z<\xi<1, \qquad \ldots(2.8)$$

$$\eta(t)\bar{\psi}_1(t) = G(t) - \eta(t)\,R(t), \quad -1 < t < y, \qquad \ldots(2.9)$$

$$R(t) = \frac{\sin(1-\rho)\pi}{\pi}(z-r)^{\rho}\int_{z}^{1}\frac{\psi_2(t)\,dt}{(t-r)(t-z)^{\rho}} \qquad \ldots(2.10)$$

and finally

$$\eta(t)\,\bar{\psi}_2(t) = T(t) - \int_{z}^{1} N(t, Q)\bar{\psi}_2(Q)\,dQ, \quad z < t < 1$$

$$T(t) = \frac{\sin(1+\alpha+\gamma-\rho)\pi}{\pi\, a_n^*}\frac{d}{dt}\int_{z}^{t}\frac{h_1(x)\,dx}{(t-x)^{\gamma+\rho-\alpha}}$$

$$-\frac{\sin(1+\alpha-\gamma-\rho)\pi}{\pi(t-z)^{\gamma+\rho-\alpha}}\int_{-1}^{y}\frac{(z-r)^{\gamma+\rho-\alpha}}{(t-r)}\,G_1(r)\,dr \qquad \ldots(2.11)$$

$$N(t, Q) = \frac{\sin(1-\rho)\pi\sin(1+\alpha-\gamma-\rho)\pi}{\pi^2(t-z)^{\gamma+\rho-\alpha}(Q-z)}$$

$$-\int_{y}^{z}\frac{(z-r)^{\gamma+2\rho-\alpha}\,\eta(r)\,dr}{(t-r)(Q-r)}. \qquad \ldots(2.12)$$

This solution is also valid under essentially the same conditions.

The above equation (2.11) is a Fredholm integral equation of second kind in the unknown function $\bar{\psi}_2$.

3. Particular Cases

(i) When $l = 0$, $\lambda = \alpha + \beta - \mu$, $\gamma = \alpha + \beta - \delta$, $\delta = \beta$, $\mu = \mu' - \beta$ and $\mu' = \beta + 1/2$, the triple series eqns. (1.1) to (1.4) would correspond to Srivastava's (1969) eqns. (1.2) to (1.7) and solutions derived here are in complete agreement to those solutions. Further, if $\alpha = \beta = 0$, we get Collins (1962) results.

(ii) If we put $y = -1$, the triple series equations (1.1) and (1.2) reduce to the dual series equations considered recently by Srivastava (1979, 1980) with $g(x) = 0$ and it is easily shown that the above solution reduces to the solution obtained by Srivastava. Further if we let $z \to 1$ in equations (1.3) and (1.4), they reduce to the dual series equations considered by Srivastava (1979, 1980) with $f(x) = 0$ and the above solution can be shown to agree with the solution obtained by him.

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