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SPHERICALLY SYMMETRIC CONFORMALLY-FLAT PERFECT FLUID DISTRIBUTIONS OF CLASS-1

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It is well-known that the Schwarzschild interior solution is the most general static spherically symmetric perfect fluid distribution which is of class-1 as well as, conformal to a flat space-time. Herein we have obtained the most general non-static perfect fluid distribution which is conformally flat as well as of class-1.

1. INTRODUCTION

In general relativity the mechanical behaviour of the space-time manifold is characterised by the energy momentum tensor T_i^j defined by the Einstein field equations

$$-8\pi T_i^j = R_j^j - \frac{1}{2} \delta_i^j R. \quad ...(1.1)$$

In particular a perfect fluid distribution of matter is characterised by the equation

$$T_i^j = (p+\rho) v_i v^j - p\delta_i^j \quad ...(1.2)$$

where p and ρ describe the pressure and density of the fluid and v^i is a time-like flow-vector.

A space-time manifold is said to be of class-1 if there exists a symmetric tensor b_{ij} satisfying the conditions

$$R_{hijk} = e(b_{hj} b_{ik} - b_{hk} b_{ij}), \; e = \pm 1 \quad ...(1.3)$$

$$b_{ij;k} - b_{ik;j} = 0 \quad ...(1.4)$$

where a semicolon (;) denotes covariant differentiation (Eisenhart 1966).

It is well-known (Pandey and Gupta 1970) that a perfect fluid distribution of class-1 is of two distinct types. One type of perfect fluid continuum is conformal to a flat space-time while the other type is not. In this paper we consider a spherically symmetric space-time in the form

$$ds^2 = - e^{\lambda} dr^2 - r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) + e^{\nu} dt^2 \quad ...(1.5)$$

where λ and ν are functions of r and t, and investigate those perfect fluid distributions which are of class-1 as well as conformal to a flat space-time. In so far as the static case is concerned it is well-known that the Schwarzschild interior solution is the only solution which is of class-1 as well as conformal to a flat space-time (Singh 1955).

Herein we have obtained the most general conformally flat perfect fluid distributions of class-1 compatible with the space-time described by (1.5). The metric potentials are obtained as functions involving density and pressure and relations have been obtained for determining pressure and density.

It is worthwhile mentioning here that the method adopted here leads us to the solutions already obtained by Vaidya (1968). The main assumption in Vaidya's investigation is that the stream worldlines of the fluid distribution are normal to the hypersurfaces ρ = constant, whereas in this paper it is the class-1 and conformal flatness condition that leads to the desired solution.

2. Class-1 Conformally-Flat Spherically-Symmetric Perfect-Fluid Distributions

We consider a spherically symmetric line element in orthogonal form

$$ds^2 = -e^{\lambda} dr^2 - r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) + e^{\nu} dt^2 \qquad \ldots(2.1)$$

where

$$\lambda = \lambda(r,t) \text{ and } \nu = \nu(r,t).$$

The nonvanishing components of T_i^j for the line-element (2.1) are given below

$$\left.\begin{aligned}
8\pi T_1^1 &= -\frac{\nu'}{r} e^{-\lambda} + \frac{1}{r^2}(1 - e^{-\lambda}) \\
8\pi T_2^2 &= \left(\frac{\ddot{\lambda}}{2} - \frac{\dot{\lambda}\dot{\nu}}{4} + \frac{\dot{\lambda}^2}{4}\right) e^{-\nu} - \left(\frac{\nu''}{2} - \frac{\lambda'\nu'}{4} + \frac{\nu'^2}{4}\right. \\
&\quad \left. + \frac{\nu' - \lambda'}{2r}\right) e^{-\lambda} \\
8\pi T_3^3 &= 8\pi T_2^2 \quad 8\pi T_4^4 = \frac{\lambda'}{r} e^{-\lambda} + \frac{1}{r^2}(1 - e^{-\lambda}) \\
8\pi T_1^4 &= \frac{\dot{\lambda}}{r} e^{-\nu}, \quad 8\pi T_4^1 = -\frac{\dot{\lambda}}{r} e^{-\lambda}.
\end{aligned}\right\} \qquad \ldots(2.2)$$

The over head dashes and dots indicate partial differentiation with respect to r and t respectively.

Now for a perfect fluid distribution the energy momentum tensor T_i^j is of the form

$$8\pi T_i^j = (a + b) \, v_i v^j - b \, \delta_i^j \qquad \ldots(2.3)$$

where $a = 8\pi\rho$, $b = 8\pi p$, $g_{ij} v^i v^j = 1$.

Here p, ρ and v^i are pressure, density, and flow vector respectively, characterising the physical features of the distribution.

In view of (2.2), the perfect fluid conditions (2.3) reduce to

$$-\frac{\nu'}{r} e^{-\lambda} + \frac{1}{r^2}(1 - e^{-\lambda}) = (a + b) \, v_1 v^1 - b \qquad \ldots(2.4)$$

$$\left(\frac{\ddot{\lambda}}{2} - \frac{\dot{\lambda}\dot{\nu}}{4} + \frac{\dot{\lambda}^2}{4}\right) e^{-\nu} - \left(\frac{\nu''}{2} - \frac{\lambda'\nu'}{4} + \frac{\nu'^2}{4} + \frac{\nu' - \lambda'}{2r}\right) e^{-\lambda} = -b \quad ...(2.5)$$

$$\lambda' r^{-1} e^{-\lambda} + r^{-2}(1 - e^{-\lambda}) = (a+b)\, v_4 v^4 - b \quad ...(2.6)$$

$$\dot{\lambda} r^{-1} e^{-\lambda} = -(a+b)\, v^1 v_4 \quad ...(2.7)$$

$$v_1 v^1 + v_4 v^4 = 1, \; v^2 = 0, \; v^3 = 0. \quad ...(2.8)$$

Further we also know that the metric (2.1) will be of imbedding class-1 if and only if it satisfies Karmarkar's condition (1948), given by

$$3F^2 + 8\pi F\left(4T^2_2 - T^1_1 - T^4_4\right) + 64\pi^2\left(T^4_1\, T^1_4 - T^1_1\, T^4_4\right) = 0 \quad ...(2.9)$$

where

$$F = r^{-2}(1 - e^{-\lambda}). \quad ...(2.10)$$

Now in view of (2.3), the condition (2.9) leads to

$$(3F - a)(F - b) = 0. \quad ...(2.11)$$

The first factor equated to zero gives a space-time with vanishing conformal curvature, while the second factor equated to zero gives a space-time with non-vanishing conformal curvature. We deal here the first case which in view of (2.10) leads to

$$e^{-\lambda} = 1 - \tfrac{1}{3}ar^2. \quad ...(2.12)$$

Now in view of (2.12), the eqns. (2.4), (2.6) and (2.7), reduce to

$$(a+b)\, v_1 v^1 = \frac{a}{3} + b - \nu' r^{-1}\left(1 - \frac{ar^2}{3}\right) \quad ...(2.13)$$

$$(a+b)\, v_4 v^4 = a + b + \tfrac{1}{3} a' r \quad ...(2.14)$$

and

$$(a+b)\, v_4 v^1 = -\tfrac{1}{3}\dot{a} r \quad ...(2.15)$$

respectively.

From (2.13), (2.14) and (2.8) one gets,

$$\nu' = r\left(1 - \tfrac{1}{3}ar^2\right)^{-1}\left(b + \tfrac{1}{3}a + \tfrac{1}{3}a'r\right). \quad ...(2.16)$$

Dividing (2.13) by (2.15) and eliminating ν' with the help of (2.16) we find,

$$\dot{a}\, v_1 - a' v_4 = 0. \quad ...(2.17)$$

Similarly (2.14) and (2.15) gives

$$3\left(a + b + \tfrac{1}{3}a'r\right) v^1 + \dot{a}\, r\, v^4 = 0. \quad ...(2.18)$$

The equation (2.18) can be rewritten as,

$$-\left(a + b + \tfrac{1}{3}a'r\right) e^{-\lambda} v_1 + \tfrac{1}{3}\dot{a} r\, e^{-\nu} v_4 = 0. \quad ...(2.19)$$

For a nontrivial solution of the system (2.17) and (2.19), we require

$$\frac{r\dot{a}^2}{3} \quad e^{-\nu} - a'(a + b + \tfrac{1}{3} a'r) e^{-\lambda} = 0 \qquad \ldots(2.20)$$

which gives e^{ν} in the form,

$$e^{\nu} = \frac{r\dot{a}^2}{3a'} \left(1 - \frac{1}{3} a'r^2\right)^{-1} \left(a + b + \tfrac{1}{3} a'r\right)^{-1}. \qquad \ldots(2.21)$$

Consistency of (2.16) and (2.21) demands that,

$$\frac{a''}{a'} - \frac{2\dot{a}'}{\dot{a}} - \frac{1}{r} + \frac{r}{3}(3b - a)\left(1 - \frac{ar^2}{3}\right)^{-1} + \left(\frac{a''r}{3}\right.$$

$$\left. + \frac{4a'}{3} + b'\right)\left(a + b + \frac{a'r}{3}\right)^{-1} = 0. \qquad \ldots(2.22)$$

In view of (2.13), (2.16) and 2.17), we find that the nonvanishing components of flow-vector are determined interms of a and b as

$$v_1 = \left(\tfrac{1}{3} ra'\right)^{\frac{1}{2}} \left(1 - \tfrac{1}{3}ar^2\right)^{-\frac{1}{2}} (a + b)^{-\frac{1}{2}} \qquad \ldots(2.23)$$

$$v_4 = \frac{\dot{a}}{a'} v_1. \qquad \ldots(2.24)$$

Further, in view of (2.12), (2.16) and (2.21) eqn. (2.5) leads to

$$\frac{\dot{a}'}{\dot{a}}(2a'r + 3a + 3b) - 2a''r + \frac{3}{\dot{a}}(\dot{b}a' - \dot{a}a' - 2\dot{a}b')$$

$$+ \frac{r}{3}\left\{2a'r(a'r + 3a + 3b) - (3b + a + a'r)(3a + 3b + 2a'r)\right\}$$

$$\times \left(1 - \frac{1}{3} ar^2\right)^{-1} = 0. \qquad \ldots(2.25)$$

Addition of (2.25) and (2.22), yields

$$\frac{a''}{a'} - \frac{\dot{a}'}{\dot{a}} - \frac{\dot{a}b' - a'\dot{b}}{\dot{a}(a + b)} - \left(\frac{1}{r} + \frac{ar}{3}\right)\left(1 - \frac{ar^2}{3}\right)^{-1} = 0. \qquad \ldots(2.26)$$

Equation (2.26), which follows from (2 22) and (2.25) has a deeper physical meaning. In fact it is the usual equation of continuity for this distribution. This has actually been verified by considering the equation of continuity, namely

$$a' v^1 + \dot{a} v^4 + (a + b)(v^1{}_{;1} + v^4{}_{;4}) = 0 \qquad \ldots(2.27)$$

where a semicolon (;) denotes covariant differentiation.

The two independent equations (2.22) and (2.26), in principle, determine a and b and the metric potentials e^{λ} and e^{ν} are determined through (2.12) and (2.21) while the two nonzero components v_1 and v_4 of the flow vector can be obtained from (2.23) and (2.24).

However, eqn. (2.26), in general, yields the solution

$$a' = r\,(a + b\,\left(1 - \tfrac{1}{3}\,ar^2\right)^{-1}\,\psi\,(a) \qquad \ldots(2.28)$$

where $\psi\,(a)$ is an arbitrary function of a.

In view of (2.28), eqn. (2.22) reduces to

$$6\dot{b}' + 2a' - \frac{6a'\dot{b}}{\dot{a}} - \frac{2a'r}{\dot{a}\,(a+b)}\,(a'\dot{b} - \dot{a}b') + \frac{r\,(a + 3b)\,(a+b)}{(1 - \frac{1}{3}\,ar^2)}$$

$$- \frac{3\,\bar{\psi}\,a'}{\psi}\,(a+b) = 0. \qquad \ldots(2.29)$$

Now treating b as a function of r and a, and denoting the partial derivative of b with respect to r, when a is treated as a constant, by b_1 and the partial derivative with respect to a by an overhead bar, the eqn. (2.29) yields

$$6b_1 + 2a' + \frac{2ra'\,b_1}{(a+b)} + \frac{r\,(a + 3b)\,(a+b)}{(1 - \frac{1}{3}\,ar^3)} - \frac{3\,\bar{\psi}\,a'}{\psi}\,(a+b) = 0. \qquad (2.30)$$

By the use of (2.28) again, the eqn. (2.30) further reduces to

$$b_1\,(3 - ar^2 + \psi\,r^2) + (a+b)\left(\psi + \frac{a}{2} + \frac{3b}{2}\right)r$$

$$- \frac{3}{2}\,\bar{\psi}\,r\,(a+b)^2 = 0. \qquad \ldots(2.31)$$

Equation (2.31) now integrates to

$$(a+b)^{-1} = \frac{3\,(1 - \bar{\psi})}{2\,(a - \psi)} - \left\{1 - \left(\frac{a - \psi}{3}\right) r^2\right\}^{\frac{1}{2}} g\,(a) \qquad \ldots(2.32)$$

where $g\,(a)$ is an arbitrary function of a and we have assumed that $\psi\,(a) \neq$ a.

3. Conclusion

Thus we have established that the most general perfect fluid distribution which is of class-1 as well as conformally flat is described by the line-element (2.1) if

$$\left.\begin{aligned} e^{\lambda} &= \left(1 - \frac{1}{3}\,ar^2\right)^{-1} \\ e^{\nu} &= r\dot{a}^2/3a'\,\left(1 - \frac{1}{3}\,ar^2\right)^{-1}\left(a + b + \frac{1}{3}\,a'r\right)^{-1} \end{aligned}\right\}. \qquad \ldots(3.1)$$

The pressure b is given by (2.32) and the density function $a \equiv a\,(r,t)$ is a solution of the diffenrential equation

$$\frac{\partial a}{\partial r} = r\,(a+b)\left(1 - \tfrac{1}{3}\,ar^2\right)^{-1}\psi\,(a). \qquad \ldots(3.2)$$

It may be recalled here that the Schwarschild interior solution is the most general static spherically symmetric perfect fluid distribution which is of class-one as well as conformal to a flat space-time. Thus the solution presented in this paper is a natural non-static generalisation of the static spherically symmetric Schwarzschild interior solution.

Vaidya (1968), in his investigations of non-static analogues of Schwarzschild's interior solution has utilised the condition that the flow-lines of the fluid be normal to the hyper surface $a(r, t) =$ constant. The continuity equation and two equations of motion together with the field equations (2.2) and (2.3) then lead to the desired solutions.

In our approach we have not used the continuity equation and the equations of motion. We have rather demanded that the space-time (2.1) be of class-one and conformal to a flat space-time. Then the field equations for a perfect fluid distribution of matter alone determine Vaidya's solutions. However the flow lines of the perfect fluid distribution turnout to be normal to the hypersurface $a(r,t) =$ constant as a consequence of the space-time being conformal to a flat space-time and of class-one.

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