Indian J. pure appl. Math., **13**(1) : 8–12, January 1982

QUASINORMAL COMPOSITION OPERATORS

R. K. SINGH AND B. S. KOMAL

Department of Mathematics, University of Jammu, Jammu 180001

(*Received* 9 *July* 1980; *after revision* 21 *July* 1981)

For bounded operators P, Q on a Hilbert space H, let $[P, Q] = PQ - QP$. Let $\theta = \{A : [A^*A, A + A^*] = 0\}$. Then it is proved that θ does not contain any non-quasinormal composition operator. It is also shown that C_T^* is an isometry if and only if it is quasinormal if and only if it is hyponormal.

1. PRELIMINARIES

Let (X, s, λ) be a σ-finite measure space and let T be a non-singular measurable transformation from X into itself. Then the composition transformation C_T from $L^p(\lambda)$ into the space of all complex valued functions on X is defined by

$$C_T f = f \circ T \text{ for every } f \in L^p(\lambda).$$

If C_T is a bounded operator and the range of C_T is contained in $L^p(\lambda)$, then we call it a composition operator induced by T.

If $X = N$, the set of all non-zero positive integers and λ is the counting measure on the family of all subsets of N, then for $p = 2$, $L^p(\lambda)$ becomes the Hilbert space of all square-summable sequences of complex numbers. This Hilbert space is denoted by l^2. The Banach algebra of all bounded linear operators on l^2 is denoted by $B(l^2)$. Let $e^{(n)}$ be the sequence defined by $e^{(n)}(p) = \delta_{np}$, the Kronecker delta. A bounded linear operator A on a Hilbert space is called quasinormal if A commutes with A^*A. If $C_T \in B(l^2)$, then C_T^*, the adjoint of C_T is computed by Singh and Komal (1978). It is given by

$$(C_T^* f)(n) = \begin{cases} \sum\limits_{m \in T^{-1}(\{n\})} f(m) & \text{if } T^{-1}(\{n\}) \text{ is non-empty}, \\ 0 & \text{if } T^{-1}(\{n\}) = \phi. \end{cases}$$

For bounded operators P and Q on a Hilbert space H, let $[P, Q] = PQ - QP$. Let $\theta = \{A : [A^*A, A + A^*] = 0\}$. The defining condition for θ appears in the work of Embry (1966). In this note we have investigated that θ does not contain any non-quasinormal composition operator. It is shown that C_T^* is quasinormal if and only if C_T^* is an isometry if and only if C_T^* is hyponormal. It is proved that the product of quasinormal composition operators need not be a quasinormal operator. But the product of their quasinormal adjoint operators is a quasinormal operator. An example is used to show that the converse to a result of Arun Bala (1977) is not true.

2. Quasinormal Composition Operators

Theorem 2.1 — Let $C_T \in B(l^2)$. Then C_T is quasinormal if and only if $C_T \in \theta$.

PROOF : The necessity is true for any bounded operator A. For the sufficiency assume that $C_T \in \theta$.

Then

$$C_T^* C_T(C_T + C_T^*) = (C_T + C_T^*) C_T^* C_T.$$

From Theorem 3 of Singh (1974),

$$M_{f_0}(C_T + C_T^*) = (C_T + C_T^*) M_{f_0}$$

or

$$M_{f_0}C_T + M_{f_0}C_T^* = C_T M_{f_0} + C_T^* M_{f_0}$$

or

$$M_{f_0}C_T - C_T M_{f_0} = C_T^* M_{f_0} - M_{f_0}C_T^*.$$

Let $n \in N$. Then

$$M_{f_0} C_T e^{(n)} - C_T M_{f_0} e^{(n)} = C_T^* M_{f_0} e^{(n)} - M_{f_0} C_T^* e^{(n)}$$

or

$$f_0 \cdot X_{\{T^{-1}(\{n\})\}} - f_0(n) X_{\{T^{-1}(\{n\})\}} = f_0(n) e^{(T(n))} - f_0 e^{(T(n))}. \tag{A}$$

Let $m \in N$ be such that $m = T(n)$. Then either $m \in T^{-1}(\{n\})$ or $m \notin T^{-1}(\{n\})$. Taking the values of functions in the equation (A) at m, we have

$$f_0(n) = f_0(T(n)).$$

Since n is arbitrary,

$$f_0 = f_0 \circ T.$$

Hence by Lemma 3 of Whitley (1978) C_T is quasinormal.

Theorem 2.2 — Let $C_T \in B(l^2)$. Then the following are equivalent: (i) C_T^* is an isometry, (ii) C_T^* is quasinormal, (iii) C_T^* is hyponormal.

PROOF : The implications (i) $=>$ (ii) and (ii) $=>$ (iii) are true for any bounded operator A. We have to show that (iii) $=>$ (i). Let $\| C_T f \| \leqslant \| C_T^* f \|$ for every $f \in l^2$. Take $n \in N$. Then

$$\| C_T^* e^{(n)} \| = \| e^{(T(n))} \| = 1$$

and

$$\| C_T e^{(n)} \|^2 = \| X_{T^{-1}(\{n\})} \|^2 = \lambda(T^{-1}(\{n\})).$$

Hence $\lambda\, T^{-1}(\{n\}) \leqslant 1$ for every $n \in N$. This shows that T is injective and hence C_T is a surjective operator. Thus by Theorem 3.1 of Singh and Komal (1978) C_T^* is an isometry. This completes the proof of the theorem.

The following example shows that the product of quasinormal composition operators need not be a quasinormal operator.

Example 2.3 — Let $T_1 : N \to N$ and $T_2 : N \to N$ be the mappings defined by

$$T_1(n) = \begin{cases} n + 1 \text{ if } n \text{ is an odd integer} \\ n - 1 \text{ if } n \text{ is an even integer} \end{cases}$$

and

$$T_2(n) = \begin{cases} n \text{ if } n \text{ is an odd integer} \\ p \text{ if } n = 2p \text{ where } p \text{ is an even integer} \\ p + 1 \text{ if } n = 2p \text{ where } p \text{ is an odd integer.} \end{cases}$$

Since T_1 is invertible, C_{T_1} is invertible (see Singh and Komal 1978) and hence by Theorem 2.3 of Singh and Komal (1978), C_{T_1} is unitary. Now $f_0(n) = f_0(T_1(n))$ and $g_0(n) = g_0(T_2(n))$ where $f_0 = (d\lambda T_1^{-1})/d\lambda$, the Radon-Nikodym derivative of the measure λT_1^{-1} with respect to λ and $g_0 = (d\lambda T_2^{-1})/d\lambda$. By Lemma 3 of Whitley (1978), C_{T_1} and C_{T_2} are quasinormal operators. Now we show that $C_{T_1} C_{T_2}$ is not a quasinormal operator. Assume $C_{T_1} C_{T_2}$ to be a quasinormal operator. Then

$$(C_{T_1} C_{T_2})^* (C_{T_1} C_{T_2}) (C_{T_1} C_{T_2}) = (C_{T_1} C_{T_2}) (C_{T_1} C_{T_2})^* (C_{T_1} C_{T_2})$$

or

$$C_{T_2}^* C_{T_1}^* C_{T_1} C_{T_2} C_{T_1} C_{T_2} = C_{T_1} C_{T_2} C_{T_2}^* C_{T_1}^* C_{T_1} C_{T_2}$$

or

$$C_{T_2}^* C_{T_2} C_{T_1} C_{T_2} = C_{T_1} C_{T_2} C_{T_2}^* C_{T_2}$$

or

$$M_{g_0} C_{T_1} C_{T_2} = C_{T_1} C_{T_2} M_{g_0} .$$

Equivalently,

$$C^*_{T_2} C^*_{T_1} M_{g_0} = M_{g_0} C^*_{T_2} C^*_{T_1}.$$

Considering the values of the operators at $e^{(1)}$, we have

$$g_0(1)\, e^{(T_2(T_1(1)))} = g_0(T_2(T_1(1)))\, e^{(T_2(T_1(1)))}.$$

This implies that $g_0(1) = g_0(T_2(T_1(1))) = g_0(T_1(1)) = 2$. This contradicts our assumption. Hence $C_{T_1} C_{T_2}$ is not a quasinormal composition operator.

Arun Bala (1977) has proved that if A_1 and A_2 are quasinormal operators and if the following conditions are satisfied

$$\text{(i)} \quad A_1A_2 = A_2A_1 \text{ and (ii) } A_1A_2^* = A_2^* A_1$$

then A_1A_2 is quasinormal. We give an example to show that the converse of the above result is not true. We first state and prove the following result:

Theorem 2.4 — Let C_{T_1} and $C_{T_2} \in B(l^2)$. Then $C^*_{T_1} C^*_{T_2}$ is a quasinormal operator if $C^*_{T_1}$ and $C^*_{T_2}$ are quasinormal operators.

PROOF : Let $C^*_{T_1}$ and $C^*_{T_2}$ be quasinormal operators. Then by the proof of Theorem 2.2 T_1 and T_2 are injections. Since $T_1 \circ T_2$ is an injection, $C_{T_1 \circ T_2}$ is a surjective operator. From Theorem 3.1 of Singh and Komal (1978) $C^*_{T_1 \circ T_2}$ is an isometry and hence it is quasinormal. Thus

$$C^*_{T_1} C^*_{T_2} = (C_{T_2} C_{T_1})^* = C^*_{T_1 \circ T_2}$$

is quasinormal.

Now we give the required example.

Example 2.5 — For $i = 1, 2$, let $T_i : N \to N$ be the mappings defined by $T_i(n) = (n + 1)\, i$ for all $n \in N$. Then $T_2(T_1(1)) = T_2(2) = 6$ and $T_1(T_2(1)) = T_1(4) = 5$. Since

$$C^*_{T_1} C^*_{T_2} e^{(1)} = C^*_{T_1} e^{(T_2(1))} = e^{(T_1(T_2(1)))}$$

$$= e^{(5)}$$

and

$$C^*_{T_2} C^*_{T_1} e^{(1)} = e^{(T_2(T_1(1)))} = e^{(6)},$$

it follows that

$$C^*_{T_1} C^*_{T_2} \neq C^*_{T_2} C^*_{T_1}.$$

Further, since

$$C^*_{T_2} C_{T_1} e^{(1)} = 0$$

and

$$C_{T_1} C^*_{T_2} e^{(1)} = C_{T_1} e^{(T_2(1))} = C_{T_1} e^{(4)} = e^{(3)},$$

we can conclude that $C_{T_1} C^*_{T_2} \neq C^*_{T_2} C_{T_1}$.

Now $C^*_{T_1}$ and $C^*_{T_2}$ are quasinormal operators. Hence by the previous theorem $C^*_{T_1} C^*_{T_2}$ is a quasinormal operator.

References

Arun, Bala (1977). A note on quasinormal operators. *Indian J. pure appl. Math.*, **8**, 463–65.

Embry, Mary (1966). Conditions implying normality in Hilbert space. *Pacific J. Math.*, **18**, 457–60.

Singh, R. K. (1974). Compact and quasinormal composition operators. *Proc. Am. Math. Soc.*, **45**, 80–82.

Singh, R. K., and Komal, B. S. (1978). Composition operator on l^p and its adjoint. *Proc. Am. Math. Soc.*, **70**, 21–25.

Whitely, Robert (1978). Normal and quasinormal composition operators. *Proc. Am. Math. Soc.*, **70**, 114–18.