A COUPLED THERMO-ELASTIC PROBLEM OF A HALF-SPACE UNDER THE ACTION OF A THERMAL SHOCK ON THE BOUNDING SURFACE

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The theory of coupled thermo-elasticity is applied to solve the problem of determination of the distribution of temperature and the thermo-elastic deformation in a half-space under the action of a thermal shock on the bounding surface. The perturbation method is used. The perturbation function for temperature is seen to vanish initially. The surface displacement has been calculated for small values of time and also a graphical representation is presented.

Introduction

Nariboli and Nyayadhish (1963) have determined the distribution of temperature and thermo-elastic deformation in a half-space, the plane boundary of which is held rigidly fixed and subjected to an instantaneous temperature rise. Solutions have been obtained by the authors for small values of time. In the present paper, the plane boundary of the half-space is free of stress and is subjected to a thermal shock. Moreover, the perturbation method is employed with the thermo-elastic coupling factor ϵ as the perturbation parameter. The solution is obtained by the application of Laplace transform. In order to find the inverse Laplace transform for the temperature, we have assumed further the thermo-elastic coupling factor to be very small. The deformation field is, however, obtained for small values of time. It may be mentioned here that the problem with the formulation of a different type of thermal boundary condition was solved by Paria (1968).

Formulation of the Problem: Governing Equations

We consider an elastic half-space D: $x \geqq 0$ with the surface plane $x = 0$ assumed free of tractions for all time. The solid is assumed to be mechanically constrained so that the displacements are of the form $u_x = u(x, t)$, $u_y = u_z = 0$ and the temperature distribution is of the form $T = T(x, t)$, x and t denoting respectively the space-coordinate and time.

The coupled thermo-elastic differential equations are (Nickell and Sackman 1968)

$$(\lambda+2\mu)\frac{\partial^2 u}{\partial x^2} = \rho\frac{\partial^2 u}{\partial t^2}+\alpha(3\lambda+2\mu)\,.\frac{\partial T}{\partial x} \qquad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$K\frac{\partial^2 T}{\partial x^2} = \rho\,.\,c_v\,.\frac{\partial T}{\partial t}+\alpha(3\lambda+2\mu)\,.\,\overset{*}{T}\frac{\partial^2 u}{\partial x\partial t} \qquad \ldots \quad \ldots \quad \ldots \quad (2)$$

where λ, μ are the isothermal Lamé constants, ρ is the density of the material of the half-space, α is the linear coefficient of thermal expansion, $\overset{*}{T}$ is the reference temperature, K is the thermal conductivity, and c_v is the specific heat at constant volume.

The initial conditions are taken to be

$$u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0.$$

The boundary condition is that the normal stress $\sigma_{xx}(0, t) = 0$. Introducing dimensionless variables

$$\left.\begin{aligned} x_1 &= \frac{ax}{K}, \qquad t_1 = \frac{a^2\,.\,t}{K}, \qquad \sigma = \frac{\sigma_{xx}}{\beta\,.\,\overset{*}{T}} \\ T_1 &= \frac{T}{\overset{*}{T}}, \qquad U_1 = \frac{a(\lambda+2\mu)}{\kappa\beta\overset{*}{T}}u \\ \kappa &= \frac{K}{\rho\,.\,C_v}, \quad a^2 = \frac{\lambda+2\mu}{\rho}, \quad \beta = (3\lambda+2\mu)\,.\,\alpha \end{aligned}\right\} \qquad \ldots \quad \ldots \quad (3)$$

eqns. (1) and (2) reduce to

$$\frac{\partial^2 U_1}{\partial x_1^2} = \frac{\partial T_1}{\partial x_1}+\frac{\partial^2 U_1}{\partial t_1^2} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

$$\frac{\partial^2 T_1}{\partial x_1^2} = \frac{\partial T_1}{\partial t_1}+\epsilon\,.\frac{\partial^2 U_1}{\partial x_1\partial t_1} \qquad \ldots \quad \ldots \quad \ldots \quad (5)$$

with the initial conditions

$$U_1,\ T_1,\ \frac{\partial U_1}{\partial t_1} = 0 \ \text{ at } t_1 = 0 \ \text{ for all } x_1 > 0, \qquad \ldots \quad \ldots \quad (6)$$

the boundary conditions

(a) $\sigma = \dfrac{\sigma_{xx}}{\beta\,.\,\overset{*}{T}} = \dfrac{\partial U_1}{\partial x_1} - T_1 = 0$ for $x_1 = 0$ $\qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7)$

for all $t_1 \geqq 0$

and

(b) $T_1 = T_0\,\delta(t_1)$

due to instantaneous thermal shock on $x_1 = 0$ where T_0 is a constant and $\delta(t_1)$ is the well-known Dirac delta function.

Besides, the regularity conditions require

$$U_1,\ T_1,\ \frac{\partial U_1}{\partial x_1},\ \frac{\partial T_1}{\partial x_1} \to 0 \ \text{ as } x_1 \to \infty. \qquad \ldots \quad \ldots \quad \ldots \quad (8)$$

The constant ϵ represents the thermo-elastic coupling factor and is given by

$$\epsilon = \frac{\beta^2 . \overset{*}{T}}{\rho . c_v(\lambda+2\mu)}.$$

Solution of the Problem: Application of Laplace Transform

We define the Laplace transforms of $U_1(x_1, t_1)$, $T_1(x_1, t_1)$ by

$$\left.\begin{aligned} \bar{U}_1(x_1, p) &= \int_0^\infty U_1(x_1, t_1) . e^{-p . t_1} \, dt_1 \\ \bar{T}_1(x_1, p) &= \int_0^\infty T_1(x_1, t_1) . e^{-p . t_1} \, dt_1 \end{aligned}\right\}. \qquad \ldots \quad \ldots \quad (9)$$

Taking the Laplace transforms of eqns. (4) and (5), we obtain

$$\frac{d^2\bar{U}_1}{dx_1^2} - p^2 . \bar{U}_1 = \frac{d\bar{T}_1}{dx_1} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

$$\frac{d^2\bar{T}_1}{dx_1^2} - p . \bar{T}_1 = \epsilon . p \frac{d\bar{U}_1}{dx_1} \qquad \ldots \quad \ldots \quad \ldots \quad (11)$$

with

$$\left.\begin{aligned} &\bar{U}_1(x_1, p), \bar{T}_1(x_1, p) \to 0 \text{ as } x_1 \to \infty, \\ &\bar{T}_1 = T_0 \quad \text{on } x_1 = 0 \\ \text{and} \quad &\frac{d\bar{U}_1}{dx_1} = T_0 \text{ on } x_1 = 0 \end{aligned}\right\}. \qquad \ldots \quad \ldots \quad (12)$$

Eliminating $\bar{T}_1$ from (10) and (11), we obtain

$$\frac{d^4\bar{U}_1}{dx_1^4} - p(1+p+\epsilon) . \frac{d^2\bar{U}_1}{dx_1^2} + p^3 . \bar{U}_1 = 0. \qquad \ldots \quad \ldots \quad (13)$$

Similarly, eliminating $\bar{U}_1$, we obtain also the same differential equation (13) in $\bar{T}_1$. Hence solutions satisfying the first conditions of (12) are

$$\left.\begin{aligned} \bar{U}_1 &= A . e^{-m_1 . x_1} + B . e^{-m_2 . x_1} \\ \bar{T}_1 &= A_1 . e^{-m_1 . x_1} + B_1 . e^{-m_2 . x_1} \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad (14)$$

where m_1^2, m_2^2 are the roots of the quadratic

$$x^4 - p(1+p+\epsilon) . x^2 + p^3 = 0. \qquad \ldots \quad \ldots \quad \ldots \quad (15)$$

Hence

$$m_1^2 + m_2^2 = p(1+p+\epsilon), \quad m_1^2 . m_2^2 = p^3$$

and

$$m_1, m_2 = \frac{\sqrt{p}}{2}(\alpha_0 \pm \beta_0),$$

$$\alpha_0 = \sqrt{1+p+\epsilon+2\sqrt{p}}, \quad \beta_0 = \sqrt{1+p+\epsilon-2\sqrt{p}}.$$

Now $\qquad (\bar{T}_1)_{x_1=0} = T_0$ gives $B_1 = T_0 - A_1$

and $$\left(\frac{d\bar{U}_1}{dx_1}-T_0\right)_{x_1=0}=0 \text{ gives } B=-\frac{T_0+A\,.\,m_1}{m_2}.$$

$$\therefore\quad \left.\begin{aligned}\bar{T}_1&=A_1\,.\,e^{-m_1.x_1}+(T_0-A_1)\,.\,e^{-m_2.x_1}\\ \bar{U}_1&=A\,.\,e^{-m_1.x_1}-\frac{T_0+Am_1}{m_2}\,.\,e^{-m_2.x_1}\end{aligned}\right\}. \qquad \ldots \quad \ldots \quad \ldots \quad (16)$$

Now substituting the solutions (16) into the differential equation (10), we obtain

$$A(m_1^2-p^2)\,.\,e^{-m_1.x_1}+\left\{\frac{T_0+Am_1}{m_2}\,.\,p^2-(T_0+Am_1)\,.\,m_2\right\}.\,e^{-m_2.x_1}$$
$$=-A_1\,.\,m_1\,.\,e^{-m_1.x_1}+(A_1-T_0)\,.\,m_2\,.\,e^{-m_2x_1}.$$

This is satisfied if

$$A_1m_1=Ap^2-A\,.\,m_1^2$$

and

$$m_2(T_0-A_1)=m_2(T_0+A\,.\,m_1)-\frac{p^2\,.\,(T_0+Am_1)}{m_2}.$$

Solving for A, A_1 we get

$$A=\frac{m_1}{m_2^2-m_1^2}T_0,\quad A_1=\frac{m_1^2-p^2}{m_1^2-m_2^2}T_0.$$

Hence

$$\bar{T}_1=T_0\left[\frac{m_1^2-p^2}{m_1^2m_2^2}\,.\,e^{-m_1.x_1}-\frac{m_2^2-p^2}{m_1^2-m_2^2}\,.\,e^{-m_2.x_1}\right] \qquad \ldots \quad \ldots \quad (17)$$

$$\bar{U}_1=T_0\left[\frac{m_2}{m_1^2-m_2^2}\,.\,e^{-m_2.x_1}-\frac{m_1}{m_1^2-m_2^2}\,.\,e^{-m_1.x_1}\right]. \qquad \ldots \quad (18)$$

Since ϵ is generally very small, we expand in ascending powers of ϵ and retain terms up to its first power.

Hence

$$\alpha_0=\sqrt{(\sqrt{p}+1)^2+\epsilon}\simeq(\sqrt{p}+1)\,.\left\{1+\frac{\epsilon}{2}\cdot\frac{1}{(\sqrt{p}+1)^2}\right\}$$
$$=(\sqrt{p}+1)+\frac{\epsilon}{2}\cdot\frac{1}{\sqrt{p}+1}.$$

Similarly,

$$\left.\begin{aligned}\beta_0&\simeq(\sqrt{p}-1)+\frac{\epsilon}{2}\cdot\frac{1}{\sqrt{p}-1}\\ m_1&\simeq p\left(1+\frac{\epsilon}{2}\cdot\frac{1}{p-1}\right)\end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad (19)$$

$$m_2\simeq\sqrt{p}\left(1-\frac{\epsilon}{2}\cdot\frac{1}{p-1}\right)$$

$$\frac{1}{m_1^2-m_2^2}\simeq\frac{1}{p(p-1)}\cdot\left\{1-\epsilon\,.\,\frac{p+1}{(p-1)^2}\right\}.$$

Also

$$m_1^2-p^2 \simeq \frac{\epsilon \,.\, p^2}{p-1}, \quad m_2^2-p^2 \simeq p\left(1-\frac{\epsilon}{p-1}-p\right).$$

Using these approximations

$$\frac{\bar{T}_1(x_1, p)}{T_0} = e^{-x_1\sqrt{p}}+\epsilon\left[\frac{e^{-p\,.\,x_1}}{p-1}+\frac{e^{-p\,.\,x_1}}{(p-1)^2}\right.$$
$$\left.+\frac{x_1}{2}\,.\,\frac{e^{-\sqrt{p}\,.\,x_1}}{(\sqrt{p}+1)}-\frac{2-x_1}{2}\,.\,\frac{e^{-x_1\sqrt{p}}}{p-1}-\frac{e^{-x_1\,.\,\sqrt{p}}}{(p-1)^2}\right] \quad \cdots \quad (20)$$

$$\frac{\bar{U}_1(x_1, p)}{T_0} = \frac{e^{-x_1\sqrt{p}}}{\sqrt{p}(p-1)}-\frac{e^{-x_1\,.\,p}}{p-1}+\epsilon\left[\frac{p+1}{p-1}\,.\,e^{-p\,.\,x_1}+\frac{x_1}{2}\,.\,\frac{p}{(p-1)^2}\,.\,e^{-p\,.\,x_1}\right.$$
$$\left.+\frac{x_1}{2}\,.\,\frac{e^{-x_1\,.\,\sqrt{p}}}{(p-1)^2}-\frac{1}{2}\,.\,\frac{e^{-x_1\sqrt{p}}}{(p-1)^2}-\frac{1}{2}\,.\,\frac{e^{-x_1\sqrt{p}}}{\sqrt{p}(p-1)^2}-\frac{p+1}{\sqrt{p}(p-1)^3}\,.\,e^{-x_1\,.\,\sqrt{p}}\right]. \quad \cdots \quad (21)$$

Temperature Field

Using the table (Erdélyi 1954), the temperature field is given by

$$\frac{T(x_1, t_1)}{T_0} = \frac{x_1}{2\sqrt{\pi t_1^3}}\,.\,e^{-\frac{x_1^2}{4t_1}}+\epsilon\left[e^{t_1-x_1}\,.\,H(t_1-x_1)+(t_1-x_1)\,.\,e^{t_1-x_1}\,.\,H(t_1-x_1)\right.$$
$$+\frac{x_1}{2}\left\{\frac{1}{\sqrt{\pi t_1}}\,.\,e^{-\frac{x_1^2}{4t_1}}-e^{x_1+t_1}\,.\,\operatorname{erf}c\left(\frac{x_1}{2\sqrt{t_1}}+\sqrt{t_1}\right)\right\}-\left(1-\frac{x_1}{2}\right).\frac{e^{t_1}}{2}$$
$$\times\left\{e^{-x_1}\,.\,\operatorname{erf}c\left(\frac{x_1}{2\sqrt{t_1}}-\sqrt{t_1}\right)+e^{x_1}\,.\,\operatorname{erf}c\left(\frac{x_1}{2\sqrt{t_1}}+\sqrt{t_1}\right)\right\}$$
$$-\frac{t_1\,.\,e^{t_1}}{2}\left\{e^{-x_1}\,.\,\operatorname{erf}c\left(\frac{x_1}{2\sqrt{t_1}}-\sqrt{t_1}\right)+e^{x_1}\,.\,\operatorname{erf}c\left(\frac{x_1}{2\sqrt{t_1}}+\sqrt{t_1}\right)\right\}$$
$$\left.+\frac{x_1}{4}\,.\,e^{t_1}\left\{e^{-x_1}\,.\,\operatorname{erf}c\left(\frac{x_1}{2\sqrt{t_1}}-\sqrt{t_1}\right)-e^{x_1}\,.\,\operatorname{erf}c\left(\frac{x_1}{2\sqrt{t_1}}+\sqrt{t_1}\right)\right\}\right]$$
$$=\frac{x_1}{2\sqrt{\pi t_1^3}}\,.\,e^{-x_1^2/4t_1}+\epsilon\,.\,F(x_1, t_1) \quad \cdots \quad (22)$$

where

$$F(x_1, t_1) = e^{t_1-x_1}\,.\,H(t_1-x_1)+(t_1-x_1)\,.\,e^{t_1-x_1}\,.\,H(t_1-x_1)$$
$$+\frac{x_1}{2}\left\{\frac{1}{\sqrt{\pi t_1}}\,e^{-x_1^2/4t_1}-\ \ldots\right.$$

In the temperature distribution (22), the first term on the right-hand side represents the solution of the classical heat conduction equation while the function $F(x_1, t_1)$ is the perturbation due to the thermo-elastic coupling coefficient ϵ. $F(x_1, t_1)$ is the perturbation function for temperature. It is seen that the perturbation function is zero when t_1 is zero for all values of $x_1 > 0$.

Deformation Field

We calculate the thermo-elastic deformation for small values of time. In this case the parameter p is large and the expansions will be in inverse powers of p. Hence using the approximations (19) and expanding for large p, keeping terms up to $p^{-7/2}$, we obtain from (21)

$$\frac{\bar{U}_1(x_1, p)}{T_0} = \left(\frac{\epsilon . x_1}{2}+2\epsilon-1\right). \frac{e^{-x_1 . p}}{p}+(x_1 . \epsilon+2\epsilon-1) . \frac{e^{-x_1 . p}}{p^2}$$
$$+\left(\frac{3\epsilon x_1}{2}+\epsilon-1\right). \frac{e^{-x_1 . p}}{p^3}+\left(x_1 . \frac{\epsilon}{2}-\frac{\epsilon}{2}\right). \frac{e^{-x_1 . \sqrt{p}}}{p^2}+(x_1 . \epsilon-\epsilon) . \frac{e^{-x_1 . \sqrt{p}}}{p^3}$$
$$+\frac{e^{-x_1 . \sqrt{p}}}{p^{3/2}}+\left(1-\frac{3\epsilon}{2}\right). \frac{e^{-x_1 . \sqrt{p}}}{p^{5/2}}+(1-5\epsilon) . \frac{e^{-x_1 . \sqrt{p}}}{p^{7/2}} .$$

Taking the inversion, we get, for small values of time, the deformation field

$$\frac{U_1(x_1, t_1)}{T_0} = \left(\frac{\epsilon . x_1}{2}+2\epsilon-1\right). H(t_1-x_1)+(x_1 . \epsilon+2\epsilon-1) . H(t_1-x_1)(t_1-x_1)$$
$$+\left(\frac{3\epsilon x_1}{2}+\epsilon-1\right). H(t_1-x_1) . \frac{(t_1-x_1)^2}{2}$$
$$+\left(x_1 . \frac{\epsilon}{2}-\frac{\epsilon}{2}\right). 4t_1 . i^2 \operatorname{erf} c\left(\frac{x_1}{2\sqrt{t_1}}\right)$$
$$+(x_1 . \epsilon-\epsilon) . (4t_1)^2 . i^4 \operatorname{erf} c\left(\frac{x_1}{2\sqrt{t_1}}\right)+2\sqrt{t_1} . i^1 \operatorname{erf} c\left(\frac{x_1}{2\sqrt{t_1}}\right)$$
$$+\left(1-3 . \frac{\epsilon}{2}\right). (4t_1)^{3/2} . i^3 \operatorname{erf} c\left(\frac{x_1}{2\sqrt{t_1}}\right)$$
$$+(1-5 . \epsilon) . (4t_1)^{5/2} . i^5 \operatorname{erf} c\left(\frac{x_1}{2\sqrt{t_1}}\right)$$

where $i^n \operatorname{erf} c(x)$ denotes the associated complementary error function of the nth degree and $H(\eta)$ is the Heaviside unit function defined by

$$H(\eta) = 1, \quad \eta > 0$$
$$= 0, \quad \eta < 0.$$

When $x_1 = 0$, the surface displacement is given by

$$\frac{U(0, t_1)}{T_0} = (2\epsilon-1) H(t_1)+(2\epsilon-1) . H(t_1) . t_1+(\epsilon-1) H(t_1) . \frac{t_1^2}{2}$$
$$-2\epsilon t_1 \times i^2 \operatorname{erf} c(0)-16t_1^2 . \epsilon \times i^4 \operatorname{erf} c(0)+2\sqrt{t_1} \times i^1 \operatorname{erf} c(0)$$
$$+\left(1-\frac{3\epsilon}{2}\right). 8 . t_1^{3/2} \times i^3 \operatorname{erf} c(0)+(1-5\epsilon) . 32t_1^{5/2} \times i^5 \operatorname{erf} c(0).$$

When the material of the half-space is copper, $\epsilon = 0{\cdot}0168$. The values of surface displacement for small values of time are given in Table I.

TABLE I

Surface displacement for small values of time

$t_1 \rightarrow$	0	0·000001	0·0001	0·001	0·01	0·04	0·09
$-\dfrac{U(0, t_1)}{T_0} \rightarrow$	0·9664	0·9653	0·9552	0·9317	0·8626	0·7738	0·6932

Surface displacement for small values of time is graphically represented in Fig. 1.

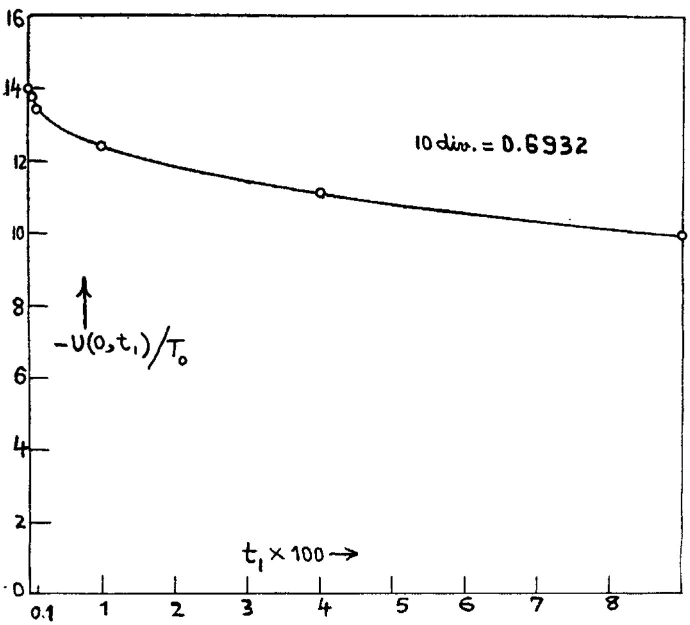

FIG. 1.

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