ON CERTAIN FINITE AND INFINITE SUMMATION FORMULAE OF GENERALIZED FOX H-FUNCTIONS

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In this paper some finite and infinite series summation of generalized Fox H-function is summed, expressing the H-function as Mellin-Barnes types of double integral and then interchanging the order of integration and summation.

1. INTRODUCTION

Fox's H-function of two variables recently introduced by Verma (1971) and Pathak (1970) is an extension of the G-function of two variables defined by Agarwal (1965) and Sharma (1965). This H-function of two variables does not only include Fox's H-function and the Meijer's G-function of single variable as a particular case but also most of special functions of two variables, e.g. Appell functions (F_1, F_2, F_3 and F_4), the Whittaker function of two variables etc. Therefore, the results established in this paper are of a general character and hence encompass several cases of interest.

Recently Sharma (in press *a*), Srivastava and Verma (1970) and Verma (1968) have summed some finite summations involving H- and G-function of two variables. The object of this paper is to establish some finite and infinite summation of generalized Fox H-function and recurrence relations based upon them.

The following notation will be used throughout this paper. The symbol (a_p, e_p) denotes the sequence of elements (a_1, e_1), (a_2, e_2), ..., (a_p, e_p), $(a_n) = \dfrac{\Gamma(a+n)}{\Gamma(a)}$, i.e. $(a)_0 = 1$, $(a)_n = a(a+1), \ldots, (a+n-1)$, $n = 1, 2, \ldots$; and symbol $\prod_{\beta,\gamma}$ indicates that the product involving β is multiplied by a similar product of γ in place of β.

Verma (1971) and Pathak (1970) defined the H-function of two variables in the form of double Mellin-Barnes type integral which has been denoted by Sharma (in press, *b*) as

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$$H^{n_1, v_1, v_2, m_1, m_2}_{p, (t:t'), s, (q:q')}\left[\begin{array}{c|l} x & (a_p, e_p) \\ & (\gamma_t, c_t); (\gamma'_{t'}, c'_{t'}) \\ y & (\delta_s, d_s) \\ & (\beta_q, b_q); (\beta'_{q'}, b'_{q'}) \end{array}\right]$$

$$= \frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \phi(\xi+\eta)\psi(\xi, \eta)x^\xi y^\eta d\xi\, d\eta, \quad \ldots \quad (1.1)$$

where

$$\phi(\xi+\eta) = \frac{\prod_{j=1}^{n_1} \Gamma(1-a_j+e_j\xi+e_j\eta)}{\prod_{j=n_1+1}^{p} \Gamma(a_j-e_j\xi-e_j\eta) \prod_{j=1}^{s} \Gamma(\delta_j+d_j\xi+d_j\eta)},$$

$$\psi(\xi, \eta) = \frac{\prod_{j=1}^{m_1} \Gamma(\beta_j-b_j\xi) \prod_{j=1}^{v_1} \Gamma(\gamma_j-c_j\xi) \prod_{j=1}^{m_2} \Gamma(\beta'_j-b'_j\eta) \prod_{j=1}^{v_2} \Gamma(\gamma'_j+c'_j\eta)}{\prod_{j=m_1+1}^{q} \Gamma(1-\beta_j+b_j\xi) \prod_{j=v_1+1}^{t} \Gamma(1-\gamma_j-c_j\xi) \prod_{j=m_2+1}^{q'} \Gamma(1-\beta'_j+b'_j\eta)}$$

$$\prod_{j=v_2+1}^{t'} \Gamma(1-\gamma'_j-c'_j\eta)$$

and $0 \leqslant m_1 \leqslant q,\ 0 \leqslant m_2 \leqslant q',\ 0 \leqslant v_1 \leqslant t,\ 0 \leqslant v_2 \leqslant t',\ 0 \leqslant n_1 \leqslant p.$

The sequence of parameters $\left(\beta_{m_1}, b_{m_1}\right), \left(\beta'_{m_2}, b'_{m_2}\right), \left(\gamma_{v_1}, c_{v_1}\right), \left(\gamma'_{v_2}, c'_{v_2}\right)$ and (a_{n_1}, e_{n_1}) are such that none of the poles of integrand coincides. The paths of integration are indented, if necessary, in such a manner that all the poles of $\Gamma(\beta_j-b_j\,\xi),\ j = 1, 2, 3, \ldots, m_1$ and $\Gamma(\beta'_k-b'_k\,\eta),\ k = 1, 2, \ldots, m_2$ lie to the right and those of $\Gamma(\gamma_j+c_j\xi),\ j = 1, 2, \ldots, \nu_1,\ \Gamma(\gamma'_k+c'_k\eta),\ k = 1, 2, \ldots, \nu_2$ and $\Gamma(1-a_j+e_j\xi+e_j\eta),\ j = 1, 2, \ldots, n_1$ lie to the left of the imaginary axis.

The integral (1.1) converges, if

$$T \equiv \sum_{j=1}^{n_1} e_j + \sum_{j=1}^{v_1} c_j + \sum_{j=1}^{m_1} b_j - \sum_{j=n_1+1}^{p} e_j - \sum_{j=1}^{s} d_j - \sum_{j=v_1+1}^{t} c_{j'} - \sum_{j=m_1+1}^{q'} b_j > 0.$$

$$|\arg x| < \tfrac{1}{2} T\pi,$$

$$T' \equiv \sum_{j=1}^{n_1} e_j + \sum_{j=1}^{v_2} c'_j + \sum_{j=1}^{m_2} b'_j - \sum_{j=n_1+1}^{p} e_j - \sum_{j=1}^{s} d_j - \sum_{j=v_1+1}^{t'} c'_j - \sum_{j=m_1+1}^{q'} b_j > 0,$$

$$|\text{ary}| < \tfrac{1}{2} T'\pi.$$

The multiplication formula for the gamma-function is

$$\Gamma(mz) = (2\pi)^{\frac{1}{2}(1-m)}\, m^{mz-\frac{1}{2}} \prod_{i=0}^{m-1} \Gamma\left(z+\frac{i}{m}\right), \quad \ldots \quad (1.2)$$

where m is a positive integer.

From it, if n is a positive integer, the formulae

$$\prod_{i=0}^{m-1} \Gamma\left(\frac{\alpha+n+i}{m}\right) = m^{-n}(\alpha)_n \prod_{i=0}^{} \Gamma\left(\frac{\alpha+i}{m}\right) \quad \ldots \quad (1.3)$$

$$\prod_{i=0}^{m-1} \Gamma\left(\frac{\alpha-n+i}{m}\right) = \frac{(-m)^n}{(1-\alpha)_n} \prod_{i=0}^{m-1} \Gamma\left(\frac{\alpha+i}{m}\right) \quad \ldots \quad (1.4)$$

can easily be derived.

2. Infinite Summation Formulae

The infinite summation formulae to be established are

(i) $$\sum_{n=0}^{\infty} \frac{(-1)^n(\alpha)_n(\frac{1}{2}+\alpha)_n}{n\,!\,(\frac{1}{2}\alpha)_n} H^{n_1, v_1, v_2, m_1, m_2}_{p, (t+4m : t'), s, (q, : q')}$$

$$\left[\begin{array}{c|l} x & (a_t, e_p) \\ & (\gamma_t, c_t), (\Delta(m, -\alpha+\beta-n), h), (\Delta(m, -\alpha+\gamma-n), h), (\Delta(m, \beta+n), h) \\ & (\Delta(m, \gamma+n), h); (\gamma'_{t'}, c'_{t'}) \\ y & (\delta_s, d_s) \\ & (\beta_q, b_q); (\beta'_{q'}, b_{q'}) \end{array}\right]$$

$$= \frac{m^{\alpha+\frac{1}{2}}\Gamma(\frac{1}{2})}{2^{\alpha-\beta-\gamma}\Gamma(\alpha+1)} H^{n_1, v_1, v_2, m_1, m_2}_{p, (t+4m : t'), s, (q : q')}$$

$$\left[\begin{array}{c|l} 2^{2mh}x & (a_p, e_p) \\ & (\gamma_t, c_t), (\Delta(m, \beta), h), (\Delta(m, \gamma), h), (\Delta(2m, -\alpha+\beta+\gamma,), h); (\gamma'_{t'}, c'_{t'}) \\ y & (\delta_s, d_s) \\ & (\beta_q, b_q); (\beta'_{q'}, b'_{q'}) \end{array}\right], \quad \ldots \quad (2.1)$$

$\mathrm{R}(\alpha-2\beta-2\gamma) > -2;$

(ii) $$\sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{n\,!\,(\alpha+\beta+\frac{1}{2})_n} H^{n_1, v_1, v_2, m_1, m_2}_{p, (t+m : t'), (q+m : q')}$$

$$\left[\begin{array}{c|l} x & (a_p, e_p) \\ & (\gamma_t, c_t), (\Delta(m, \frac{1}{2}-\gamma-n), h); (\gamma'_{t'}, c'_{t'}) \\ y & (\delta_s, d_s) \\ & (\Delta(m, \gamma+n), h), (\beta_q, b_q); (\beta'_{q'}, b'_{q'}) \end{array}\right]$$

$$= \frac{\Gamma(\frac{1}{2})\Gamma(\alpha+\beta+\frac{1}{2})}{\Gamma(\alpha+\frac{1}{2})\Gamma(\beta+\frac{1}{2})} H^{n_1, v_1, v_2, m_1+2m, m_2}_{p, (t+2m : t') s_1(q+2m : q')}$$

$$\left[\begin{array}{c|l} x & (a_p, e_p) \\ & (\gamma_t, c_t,), (\Delta(m, \frac{1}{2}-\gamma+\alpha), h), (\Delta(m, \frac{1}{2}-\gamma+\beta), h); (\gamma'_{t'}, c'_{t'}) \\ y & (\delta_s, d_s) \\ & (\Delta(m, \gamma+\frac{1}{2}), h), (\Delta(m, \gamma-\alpha-\beta+\frac{1}{2}), h), (\beta_q, b_q); (\beta'_{q'}, b'_{q'}) \end{array}\right], \quad \ldots \quad (2.2)$$

provided m and n are positive integers,

$$h > 0,\ |\arg x| < \tfrac{1}{2}T\pi(T > 0),\ |\arg y| < \tfrac{1}{2}T'\pi(T' > 0),$$

where

$$T \equiv \sum_{j=1}^{n_1} e_j + \sum_{j=1}^{v_1} c_j + \sum_{j=1}^{m_1} b_j - \sum_{j=n_1+2}^{p} e_j - \sum_{j=1}^{s} d_j - \sum_{j=v_1+1}^{t} c_j - \sum_{j=m_1+1}^{q} b_j$$

and

$$T' \equiv \sum_{j=1}^{n_1} e_j + \sum_{j=1}^{v_2} c_j' + \sum_{j=1}^{m_2} b_j' - \sum_{j=n_1+1}^{p} e_j - \sum_{j=1}^{s} d_j - \sum_{j=v_2+1}^{t'} c_j' - \sum_{j=m_2+1}^{q'} b_j'.$$

Proof : To prove (2.1), expressing the H-function on the left-hand side by Mellin-Barnes type integral (1.1), changing the order of summation and integration as permissible by absolute convergence for $R(\alpha-2\beta-2\gamma) < -2$ and the above stated conditions, and finally using (1.3) and (1.4), the series becomes

$$\frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty} {}_4F_3\left[\begin{matrix}\alpha,\ \tfrac{1}{2}\alpha+1,\ \beta+mh\xi,\ \gamma+mh\xi \\ \tfrac{1}{2}\alpha,\ \alpha-\beta+1-mh\xi,\ \alpha-\gamma+1-mh\xi\end{matrix};\ -1\right]$$

$$\times \frac{\phi(\xi+\eta)\,\psi(\xi,\eta)x^{\xi}y^{\eta}}{\prod_{\beta,\gamma}\left[\prod_{i=0}^{m-1}\left\{\Gamma\left(\frac{\alpha-\beta+1+i}{m}-h\xi\right)\Gamma\left(\frac{1-\beta+i}{m}-h\xi\right)\right\}\right]}\,d\xi\,d\eta.$$

Now simplifying the above expression with the help of result (MacRobert 1966, p. 368) :

$${}_4F_3\left[\begin{matrix}\alpha,\ \tfrac{1}{2}\alpha+1,\ \beta,\ \gamma \\ \tfrac{1}{2}\alpha,\ \alpha-\beta+1,\ \alpha-\gamma+1\end{matrix};\ -1\right] = \frac{\Gamma(\alpha-\beta+1)\Gamma(\alpha-\gamma+1)}{\Gamma(\alpha+1)\Gamma(\alpha-\beta-\gamma+1)}, \quad \ldots \quad (2.3)$$

and (1.2), we get

$$\frac{m^{\alpha+\frac{1}{2}}\,\Gamma(\frac{1}{2})}{2^{\alpha-\beta-\gamma}\Gamma(\alpha+1)}\cdot\frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}$$

$$\times \frac{\phi(\xi+\eta)\,\psi(\xi,\eta)\,(2^{2mh}x)^{\xi}y^{\eta}}{\prod_{\beta,\gamma}\left[\prod_{i=0}^{m-1}\Gamma\left(\frac{1-\beta+i}{m}-h\xi\right)\right]\prod_{i=0}^{2m-1}\Gamma\left(\frac{\alpha-\beta-\gamma+1+i}{2m}-h\xi\right)}\,d\xi d\eta.$$

On applying (1.1), the right-hand side of (2.1) is obtained.

Similarly Equation (2.2) can be obtained by using the result given by MacRobert (1958) in place of (2.3).

3. Finite Summation Formulae

The finite summation formulae to be established are

$$\text{(i)}\ \sum_{u=0}^{n} {}^{n}C_u \frac{m^{2u}}{(1+K)_u(1-K-n)_u} H^{n_1,\,v_1,\,v_2\,m_1+m,\,m_2}_{p,\,(t+m\,:\,t'),\,s,\,(q+m\,:\,q')}$$

$$\times\left[\begin{array}{c|l} & (a_p,\,e_p) \\ x & (\gamma_t,\,c_t),\,(\Delta(m,\,1+u-\alpha),\,h);\,(\gamma'_{t'},\,c'_{t'}) \\ & (\delta_s,\,d_s) \\ y & (\Delta(m,\,\alpha+u),\,h),\,(\beta_q,\,b_q);\,(\beta'_{q'},\,b'_{q'}) \end{array}\right]$$

$$= \frac{(-1)^n m^{2n}\,\Gamma(K+1)\,\Gamma(K)}{\Gamma(K+n)\,\Gamma(K+1+n)} H^{n_1,\,v_1,\,v_2\,m_1+2m,\,m_2}_{p,\,(t+2m\,:\,t'),\,s,\,(q+2m\,:\,q')}$$

$$\times\left[\begin{array}{c|l} & (a_p,\,e_p) \\ x & (\gamma_t,\,c_t),\,(\Delta(m,\,1-\alpha-K),\,h),\,(\Delta(m,\,1-\alpha+K+n),\,h);\,(\gamma'_{t'},\,c'_{t'}) \\ & (\delta_s,\,d_s) \\ y & (\Delta(m,\,\alpha+K+n),\,h),\,(\Delta(m,\,\alpha-K),\,h),\,(\beta_q,\,b_q);\,(\beta'_{q'},\,b'_{q'}) \end{array}\right] \qquad (3.1)$$

$$\text{(ii)}\ \sum_{u=0}^{n} {}^{n}C_u (K)_u (-K+n)_u\, m^{-2u}\, H^{n_1,\,v_1\,v_2,\,m_1+m,\,m_2}_{p,\,(t+m\,:\,t'),\,s,\,(q+m\,:\,q')}$$

$$\times\left[\begin{array}{c|l} & (a_p,\,e_p) \\ x & (\gamma_t,\,c_t),\,(\Delta(m,\,1-\alpha-u),\,h);\,(\gamma'_{t'},\,c'_{t'}) \\ & (\delta_s,\,d_s) \\ y & (\Delta(m,\,\alpha-u),\,h),\,(\beta_q,\,b_q);\,(\beta'_{q'},\,b'_{q'}) \end{array}\right] = H^{n_1,\,v_1+3m,\,v_2,\,m_1\,m_2}_{p,\,(t+3m\,:\,t'),\,s,\,(q+3m\,:\,q')}$$

$$\times\left[\begin{array}{c|l} & (a_p,\,e_p) \\ x & (\Delta(m,\,1-\alpha+K),\,h),\,(\Delta(m,\,1-\alpha-n),\,h), \\ & (\Delta(m,\,1-\alpha-K+n),\,h),\,(\gamma_t,\,c_t);\,(\gamma'_{t'},\,c'_{t'}) \\ & (\delta_s,\,d_s) \\ y & (\beta_q,\,b_q),\,(\Delta(m,\,\alpha+K),\,h),\,(\Delta(m,\,\alpha-n),\,h) \\ & (\Delta(m,\,\alpha-K+n),\,h)\,(\beta'_{q'},\,b'_{q'}) \end{array}\right] \qquad \ldots\ (3.2)$$

$$\text{(iii)}\ \sum_{u=0}^{n} (-1)^u\, {}^{n}C_u \frac{(\alpha)_u(\tfrac{1}{2}\alpha+1)_u(\delta)_u(\epsilon)_u}{(\tfrac{1}{2}\alpha)_u(\alpha-\delta+1)_u(\alpha-\epsilon+1)_u(\alpha+n+1)_u}\ .$$

$$H^{n_1,\,v_1,\,v_2,\,m_1+2m,\,m_2}_{p,\,(t+2m\,:\,t'),\,s,\,(q+2m\,:\,q')} \times\left[\begin{array}{c|l} & (a_p,\,e_p) \\ x & (\gamma_t,\,c_t),\,(\Delta(m,\,-\alpha+\gamma-u),\,h), \\ & (\Delta(m,\,\gamma+u),\,h);\,(\gamma'_{t'},\,c'_{t'}) \\ & (\delta_s,\,d_s) \\ y & (\Delta(m,\,\beta+u),\,h),\,(\Delta(m,\,\beta-\alpha-u),\,h), \\ & (\beta_q,\,b_q);\,(\beta'_{q'},\,b'_{q'}) \end{array}\right]$$

$$= \frac{(\alpha+1)_n(\alpha-\beta-\gamma+1)_n}{(\alpha-\delta+1)_n(\alpha-\beta-\gamma-\delta+1)_n} H^{n_1,\,v_1,\,v_2,\,m_1+4m,\,m_2}_{p,\,(t+4m\,:\,t'),\,s,\,(q+4m\,:\,q')}$$

$$\times \left[\begin{array}{c|l} & (a_p, e_p) \\ x & (\gamma_t, c_t), (\Delta(m, -\alpha+\gamma-n), h), (\Delta(m, \gamma), h), (\Delta(m, -\alpha+\gamma+\delta), h), \\ & (\Delta(m, -\beta-\delta+\alpha+n), h); (\gamma'_{t'}, c'_{t'}), \\ & (\delta_s, d_s) \\ y & (\Delta(m, \beta), h), (\Delta(m, \beta-\alpha-n), h), (\Delta(m, \beta+\delta-\alpha), h), \\ & (\Delta(m, \alpha-\gamma-\delta+1+n), h), (\beta_q, b_q); (\beta'_{q'}, b'_{q'}) \end{array} \right] \qquad (3.3)$$

$$1+2\alpha = \beta+\gamma+\delta+\epsilon+n\ ;$$

provided m and n are positive integers,

$$h > 0,\ |\arg x| < \tfrac{1}{2} T\pi\ (T > 0),\ |\arg y| < \tfrac{1}{2} T'\pi\ (T' > 0),$$

where $T \equiv \sum_{j=1}^{n_1} e_j + \sum_{j=1}^{v_1} c_j + \sum_{j=1}^{m_1} b_j - \sum_{j=n_1+1}^{p} e_j - \sum_{j=1}^{s} d_j - \sum_{j=v_1+1}^{t} c_j - \sum_{j=m_1+1}^{q} b_j$

and $T' \equiv \sum_{j=1}^{n_1} e_j + \sum_{j=1}^{v_2} c'_j + \sum_{j=1}^{m_2} b'_j - \sum_{j=n_1+1}^{p} e_j - \sum_{j=1}^{s} d_j - \sum_{j=v_2+1}^{t'} c'_j - \sum_{j=m_2+1}^{q'} b'_j.$

Proof : To prove (3.1), expressing the H-function on the left hand side by Mellin-Barnes type integral (1.1), changing the order of summation and integration, which is justified due to the absolute convergence involved in the process and finally using (1.3) and (1.4), the series becomes

$$\frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \phi(\xi+\eta)\psi(\xi, \eta)\, {}_3F_2\left[\begin{matrix} \alpha-mh\xi, & 1-\alpha+mh\xi, & -n \\ 1+K, & 1-K-n & \end{matrix} ; 1 \right] x^\xi y^\eta\, d\xi\, d\eta. \qquad \ldots \quad \ldots \quad (3.4)$$

Now we simplify the above expression with the help of Saalschütz's theorem (MacRobert, 1966, p. 360) :

If $\rho+\sigma = \alpha+\beta+\gamma+1$ and one of the parameters α, β, γ is a negative integer, then

$${}_3F_2\left[\begin{matrix} \alpha, \beta, \gamma \\ \rho, \sigma \end{matrix} ; 1 \right] = \frac{\Gamma(\rho)\, \Gamma(\alpha-\sigma+1)\, \Gamma(\beta-\sigma+1)\, \Gamma(\gamma-\sigma+1)}{\Gamma(1-\sigma)\, \Gamma(\rho-\alpha)\, \Gamma(\rho-\beta)\, \Gamma(\rho-\gamma)}, \qquad \ldots \quad (3.5)$$

$$\frac{\Gamma(1-\alpha-n)}{\Gamma(1-\alpha)} = \frac{(-1)^n\, \Gamma(\alpha)}{\Gamma(\alpha+n)} \qquad \ldots \quad \ldots \quad (3.6)$$

and (1.2), (3.4) reduces to

$$\frac{(-1)^n m^{2n}\, \Gamma(K+1)\, \Gamma(K)}{\Gamma(K+n)\, \Gamma(K+1+n)} \times \frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \phi(\xi+\eta)\psi(\xi, \eta)$$

$$\times \frac{\prod_{i=0}^{m-1} \left\{ \Gamma\left(\frac{\alpha+K+n+i}{m} - h\xi \right) \Gamma\left(\frac{\alpha-K+i}{m} - h\xi \right) \right\}}{\prod_{i=0}^{m-1} \left\{ \Gamma\left(\frac{\alpha+K+i}{m} - h\xi \right) \Gamma\left(\frac{\alpha-K-n+i}{m} - h\xi \right) \right\}} x^\xi y^\eta d\xi\, d\eta.$$

On applying (1.1), the right-hand side of (3.1) is obtained.

Similarly, we can obtain the formula (3.2) and (3.3) by using the results of (3.5) and (MacRobert 1966, p. 371 (34)) respectively.

Setting $n = 1$, (3.1), (3.2) and (3.3) yields

$$\text{(i)}\ \frac{K(K+1)}{m^2} H^{n_1,\nu_1,\nu_2,m_1,m_2}_{p,(t:t'),s,(q:q')} \left[\begin{array}{c|l} x & (a_p, e_p) \\ & (\gamma_t, c_t); (\gamma'_{t'}, c'_{t'}) \\ y & (\delta_s, d_s) \\ & (\beta_q, b_q); (\beta'_{b'}, b'_{q'}) \end{array}\right]$$

$$+H^{n_1,\nu_1,\nu_2\,m_1+2m,\,m_2}_{p,(t+2m:t'),s,(q+2m:q')} \left[\begin{array}{c|l} x & (a_p, e_p) \\ & (\gamma_t, c_t), (\Delta(m, -\alpha-K), h), \\ & (\Delta(m, 1-\alpha+K), h); (\gamma_{t'}, c_{t'}) \\ y & (\delta_s, d_s) \\ & (\Delta(m, 1+\alpha+K), h), (\Delta(m, \alpha+K), h); \\ & (\beta_q, b_q), (\beta'_{q'}, b'_{q'}) \end{array}\right]$$

$$= H^{n_1,\nu_1,\nu_2,m_1+m,\,m_2}_{p,(t+m:t'),s,(q+m:q')} \left[\begin{array}{c|l} x & (a_p, e_p) \\ & (\gamma_t, c_t), [\Delta(m, 1-\alpha), h); (\gamma'_{t'}, c'_{t'}) \\ y & (\delta_s, d_s) \\ & (\Delta(m, 1+\alpha,) h), (\beta_q, b_q); (\beta'_{q'}, b'_{q'}) \end{array}\right] \quad \ldots \quad (3.7)$$

$$\text{(ii)}\ H^{n_1,\nu_1,\nu_2,m_1+m,m_2}_{p,(t:t'),s,(q:q')} \left[\begin{array}{c|l} x & (a_p, e_p) \\ & (\gamma_t, c_t); (\gamma'_{t'}, c'_{t'}) \\ y & (\delta_s, d_s) \\ & (\beta_q, b_q); (\beta'_{q'}, b'_{q'}) \end{array}\right]$$

$$+\frac{K(1-K)}{m^2} H^{n_1,\nu_1,\nu_2,m_1+m_2,}_{p,(t+m:t')\,s,(q+m:q')} \left[\begin{array}{c|l} x & (a_p, e_p) \\ & (\gamma_t, c_t), (\Delta(m, -\alpha-1), h); (\gamma'_{t'}, c'_{t'}) \\ y & (\delta_s, d_s) \\ & (\Delta(m, \alpha-1), h), (\beta_q, b_q); (\beta'_{q'}, b'_{q'}) \end{array}\right]$$

$$= H^{n_1,\nu_1+3m,\nu_2,m_1,m_2}_{p,(t+3m:t'),s,(q+3m:q')} \left[\begin{array}{c|l} x & (a_p, e_p) \\ & (\Delta(m, 1-\alpha+K), h), (\Delta(m, -\alpha), h), \\ & (\Delta(m, 2-\alpha-K), h), (\gamma_t, c_t); (\gamma'_{t'}, c'_{t'}) \\ & (\delta_s, d_s) \\ y & (\beta_q, b_q), (\Delta(m, \alpha+K), h), \\ & (\Delta(m, \alpha-1), h), (\Delta(m, \alpha-K+1), h); \\ & (\beta'_{q'}, b'_{q'}) \end{array}\right], \quad (3.8)$$

$$\text{(iii)}\quad H^{n_1,\,v_1,\,v_2,\,m_1+2m,\,m_2}_{p,\,(t+2m\,:\,t'),\,s,\,(q+2m\,:\,q')}\cdot\left[\begin{array}{c|l} & (a_p,\,e_p) \\ x & (\gamma_t,\,c_t),\,(\Delta(m,\,-\alpha+\gamma),\,h), \\ & (\Delta(m,\,\gamma),\,h);\,(\gamma'_{t'},\,c'_{t'}) \\ & (\delta_s,\,d_s) \\ y & (\Delta(m,\,\beta),\,h),\,(\Delta(m,\,\beta-\alpha),\,h), \\ & (\beta_q,\,b_q);\,(\beta'_{q'},\,b'_{q'}) \end{array}\right]$$

$$-\frac{2\delta\,\epsilon\,(\frac{1}{2}\alpha+1)}{(\alpha+2)(\alpha-\delta+1)(\alpha-\epsilon+1)}\,H^{n_1,\,v_1,\,v_2,\,m_1+2m,\,m_2}_{p,\,(t+2m\,:\,t'),\,s,\,(q+2m\,:\,q')}$$

$$\left[\begin{array}{c|l} & (a_p,\,e_p) \\ x & (\gamma_t,\,c_t),\,(\Delta(m,\,-\alpha+\gamma-1),\,h_j),\,(\Delta(m,\,\gamma+1),\,h);\,(\gamma'_{t'},\,c'_{t'}) \\ & (\delta_s,\,d_s) \\ y & (\Delta(m,\,\beta+1),\,h),\,(\Delta(m,\,\beta-\alpha-1),\,h),\,(\beta_q,\,b_q);\,(\beta'_{q''},\,b'_{q'}) \end{array}\right]$$

$$=\frac{(\alpha+1)(\alpha-\beta-\gamma+1)}{(\alpha-\delta+1)(\alpha-\beta-\gamma-\delta+1)}\,H^{n_1\,v_1,\,v_2,\,m_1+4m,\,m_2}_{p,\,(t+4m\,:\,t'),\,(q+4m\,:\,q')}$$

$$\times\left[\begin{array}{c|l} & (a_p,\,e_p) \\ & (\gamma_t,\,c_t),\,(\Delta(m,\,-\alpha+\gamma-1),\,h),\,(\Delta(m,\,\gamma)\;\;h),\,(\Delta(m,\,-\alpha+\gamma+\delta),\,h), \\ x & (\Delta(m,\,-\beta-\delta+\alpha+1),\,h);\,(\gamma'_{t''},\,c'_{t'}) \\ & (\delta_s,\,d_s) \\ y & (\Delta(m,\,\beta),\,h),\,(\Delta(m,\,\beta-\alpha-1),\,h),\,(\Delta(m,\,\beta+\delta-\alpha),\,h), \\ & (\Delta(m,\,\alpha-\gamma-\delta+2),\,h),\,(\beta_q,\,b_q);\,(\beta'_{q''},\,b'_{q''}). \end{array}\right] \qquad \ldots\quad (3.9)$$

$$1+2\alpha=\beta+\gamma+\delta+\epsilon+n;$$

provided m is a integer,

$$h>0,\ |\arg x|<\tfrac{1}{2}T\pi\,(T>0),\ |\arg y|<T'\pi\,(T'>0),$$

where

$$T\equiv\sum_{j=1}^{n_1}e_j+\sum_{j=1}^{v_1}c_j+\sum_{j=1}^{m_1}b_j-\sum_{j=n_1+1}^{p}e_j-\sum_{j=1}^{s}d_j-\sum_{j=v_1+1}^{t}c_j-\sum_{j=m_1+1}^{q}b_j.$$

and

$$T'\equiv\sum_{j=1}^{n_1}e_j+\sum_{j=1}^{v_2}c'_j+\sum_{j=1}^{m_2}b'_j-\sum_{j=n_1+1}^{p}e_j-\sum_{j=1}^{s}d_j-\sum_{j=v_2+1}^{t'}c'_j-\sum_{j=m_2+1}^{q'}b'_j.$$

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