FLEXURAL VIBRATIONS OF ISOTROPIC ELASTIC RECTANGULAR PLATE RESTING ON ELASTIC FOUNDATION

by B. B. Sharma, *D. A. V. Post-Graduate College, Muzaffarnagar (U.P.)*

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A numerical solution for the case of a rectangular plate resting on an elastic foundation is given by including the effects of shear and rotatory inertia. The results for fundamental and first mode of vibrations are obtained.

Nomenclature

a = length of the plate

K = a constant

E = Young's modulus of elasticity

ν = Poisson's ratio

∇^2 = Laplace two-dimensional operator

h = thickness of the plate

k = modulus of Winkler's assumption

w = component of the point in the Z direction

ρ = the density of material.

Introduction

Much work has been done on the flexural vibrations of plates and beams. Mindlin (1951) gave an equation for flexural vibration of plates. He deduced a comprehensive two-dimensional theory of flexural vibrations of plates analogous to Timoshenko's (1921) one-dimensional theory of bars.

Equation of Our Problem

The two-dimensional analog of Timoshenko's beam equation including the effects of shear and rotatory inertia is given by (Sharma 1971)

$$\left(\nabla^2 - \frac{\rho}{G'}\frac{\partial^2}{\partial t^2}\right)\left(D\nabla - \frac{\rho h^3}{12}\frac{\partial^2}{\partial t^2}\right)w + \rho h\frac{\partial^2 w}{\partial t^2} = -\left(1 - \frac{D\nabla^2}{G'h} + \frac{\rho h^2}{12G'}\frac{\partial^2}{\partial t^2}\right)kw \quad (1)$$

where $G' = \dfrac{K^2E}{2(1+\nu)}$, $D = \dfrac{Eh^3}{12(1-\nu^2)}$.

To find the numerical solution we write (1) as follows

$$\nabla^4 w - \left[\frac{2(1+\nu)}{K^2}+(1-\nu^2)\right]\frac{\rho}{E}\nabla^2\frac{\partial^2 w}{\partial t^2}+12\frac{(1-\nu^2)}{h^2}\frac{\rho}{E}\frac{\partial^2 w}{\partial t^2}+\frac{2(1+\nu)(1-\nu^2)}{K^2}$$

$$\frac{\rho^2}{E^2}\frac{\partial^4 w}{\partial t^4}+\left[12\frac{(1-\nu^2)}{Eh^3}-\nabla^2\frac{2(1+\nu)}{K^2Eh}+\frac{\rho}{E^2}\frac{2(1+\nu)(1-\nu^2)}{K^2h}\frac{\partial^2}{\partial t^2}\right]kw = 0 \quad \ldots \quad (2)$$

Taking $\nu = 0{\cdot}3$ and $K^2 = 0{\cdot}85$. The latter value is based on the formula suggested by Mindlin (1951).

Let the displacement be given by $w = W \cos pt$, where p is the frequency of vibration and W is the amplitude of deflection. Then we have

$$\frac{\partial^4 W}{\partial x^4}+2\frac{\partial^4 W}{\partial x^2\partial y^2}+\frac{\partial^4 W}{\partial y^4}-\left(-3{\cdot}96882\frac{\rho}{E}p^2+\frac{3{\cdot}05882}{Eh}k\right)\nabla^2 W$$

$$-\left(\frac{10{\cdot}92}{h^2}+\frac{2{\cdot}78353}{Eh}k\right)\frac{\rho}{E}p^2W+\frac{10{\cdot}92}{Eh^3}kW+2{\cdot}78353\frac{\rho^2}{E^2}p^4W = 0 \quad \ldots \quad (3)$$

Multiplying (3) by $p^2a^2/E|\rho$ throughout and writing in two component equations by introducing M, W the bending moment and deflection of the plate respectively, we have

$$a^2\left(\frac{\partial^2 M}{\partial^2 x^2}+\frac{\partial^2 M}{\partial y^2}\right)+\left[3{\cdot}96882(p^2a^2/E|\rho)^2-3{\cdot}05882\,p^2a^2/E|\rho\,\frac{k}{E|a}\,\frac{a}{h}\right]\frac{M}{p^2a^2/E|\rho}$$

$$+\left[10{\cdot}92\left(\frac{a}{h}\right)^2+2{\cdot}78353\frac{a}{h}\frac{k}{E|a}\right](p^2a^2/E|\rho)^2W-10{\cdot}92\left(\frac{a}{h}\right)^3\frac{k}{E|a}(p^2a^2/E|\rho)W$$

$$-2{\cdot}78353(p^2a^2/E|\rho)^3\,W = 0$$

and

$$a^2\left(\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}\right) = -M|(p^2a^2/E|\rho)$$

Now take

$$A = 10{\cdot}92\left(\frac{a}{h}\right)^2+2{\cdot}78353\,\frac{a}{h}\,k/E|a,\ B = 10{\cdot}92\left(\frac{a}{h}\right)^3\frac{k}{E|a},$$

$$C = 3{\cdot}05882\,\frac{a}{h}\,k/E|a.$$

Now replacing the differential coefficients M and W with respect to X and Y by their finite difference notations, we have the following difference equations of the differential equation.

$$a^2\left(\frac{\Delta_{xx}M}{\Delta x^2}+\frac{\Delta_{yy}M}{\Delta y^2}\right)+A(p^2a^2/E\,|\,\rho)^2W-B(p^2a^2/E\,|\,\rho)W-2{\cdot}78353(p^2a^2/E\,|\,\rho)^3W+$$
$$(3{\cdot}96882p^2a^2/E\,|\,\rho-C)M=0 \quad \ldots \quad (4)$$

and

$$a^2\left(\frac{\Delta_{xx}W}{\Delta x^2}+\frac{\Delta_{yy}}{\Delta y^2}\right)=-M\,|\,(p^2a^2/E\,|\,\rho).$$

Method of Solution

Let us consider the case of a rectangular plate resting on elastic foundation simply supported at is edges of length a and width $b=(4/5)a$. In this case the values of M and W are zero at the edges and we can make an approximation for M and W by dividing the plate into a number of 80 small square parts by taking $\Delta x=\Delta y=a/10$ as shown in Fig. 1. Since the deflections

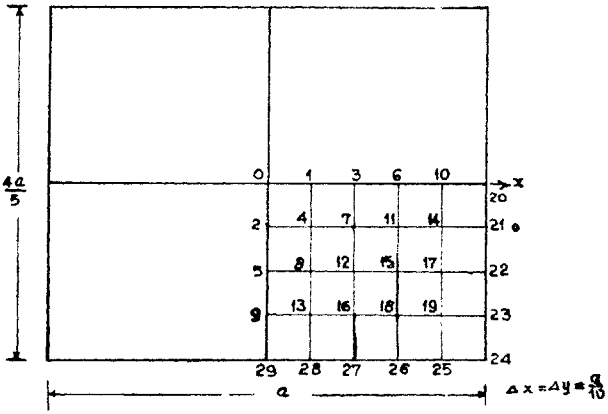

Fig. 1.

are symmetric in the four quadrants of the plate, it is enough if the calculations are extended over an area of one-fourth of the plate only. Calculating only for the interior points for which M and W are different from zero.

Denoting by M and W respectively the moment and deflection at the nth point of the portion under consideration, and taking the centre of the plate 0 (Fig. 1) the difference equation at the points 0, 1, 2, ..., 19 respectively is found by putting $n=0, 1, 2, \ldots, 19$ in terms $\left(\frac{\Delta_{xx}}{\Delta x^2}+\frac{\Delta_{yy}}{\Delta y^2}\right)$. The difference equations (Allen 1954) at the points 0, 1, 2, ..., 19 respectively are given below. Also since lower right handed quadrant of the plate is divided into 20 equal parts along the x- and y-axis. We have

$$\Delta x=\Delta y=a/10.$$

Also

$$D = [3{\cdot}96882(p^2a^2/E\,|\,\rho) - C]/100$$

$$E = [A(p^2a^2/E\,|\,\rho)^2 - B(p^2a^2/E\,|\,\rho) - 2{\cdot}78353(p^2a^2/E\,|\,\rho)^3]/100$$

$$F = \frac{1}{100}\,\frac{E\,|\,\rho}{p^2a^2}$$

$$\begin{array}{lll}
n = 0 & 2M_2+2M_1-(4-D)M_0+EW_0 & = 0 \\
 & 2W_2+2W_1-4W_0 \quad +FM_0 & = 0 \\
n = 1 & 2M_4+M_3-(4-D)M_1+M_0+EW_1 & = 0 \\
\vdots & 2W_4+W_3-4W_1 \quad +W_0+FM_1 & = 0 \\
 & \vdots & \vdots \\
n = 18 & M_{19}-(4-D)M_{18}+M_{16}+M_{15}+EW_{18} & = 0 \\
 & W_{19}-4W_{18} \quad +W_{16}+W_{15}+FM_{18} & = 0 \\
n = 19 & FR_1 = -(4-D)M_{19}+M_{18}+M_{17}+EW_{19} & = 0 \\
 & FR_2 = -4W_{19}+W_{18}+W_{17}+FM_{19} & = 0
\end{array}$$

These equations contain the values of M and W at the points $n = 0,1,2,\ldots,$ 19. Thus we have forty equations involving forty unknowns and their solutions are obtained by using matrix method with the help of I.B.M. 1620 Computer.

Procedure Followed

First we give a particular value to W and an arbitrary value to M and an approximate value to $p^2a^2/E\,|\,\rho$ given by the formula for the frequency corresponding to the fundamental and first mode of vibration. Starting with these three parameters, we successively solve these equations to get the values of M and W at the points $n = 0, 1, 2, \ldots, 19$, and having gone through the whole operation we find that these values of W's and M's do not exactly satisfy the residuals FR_1 and FR_2.

Now we keep W_0 and $p^2a^2/E\,|\,\rho$ fixed and take a slightly different value of M_0 to start with and repeat the whole process as above and we get another set of values of residuals FR_1 and FR_2.

Now again keeping W_0 and M_0 fixed and starting with a slightly different value of $p^2a^2/E\,|\,\rho$ and repeating the procedure followed above, we find another set of values of residuals FR_1 and FR_2 and see that these are not exactly zero.

With these three sets of values in which W is kept always fixed and one of the two parameters M_0 or $p^2a^2/E\,|\,\rho$ is varied we can find by interpolation new values of M_0 and $p^2a^2/E\,|\,\rho$ such that the final residuals become zero.

The above process is repeated several times with improved values of M_0 and $p^2a^2/E|\rho$ till we get residuals FR_1 and FR_2 which can be considered negligible.

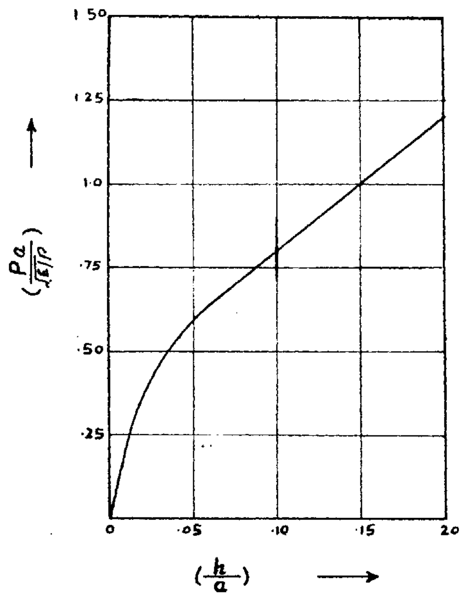

FIG. 2. Fundamental mode.

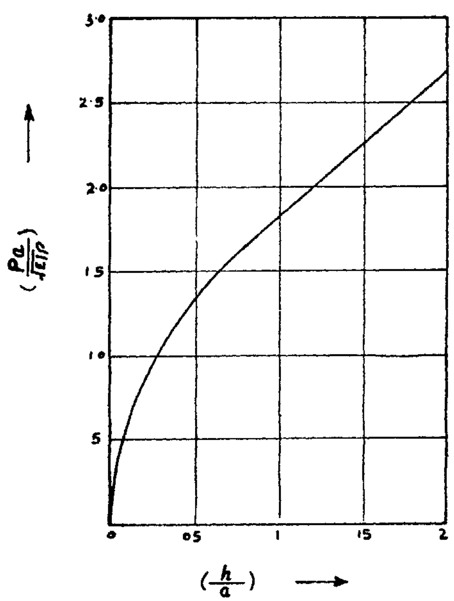

FIG. 3. First mode.

Numerical Results

Following the procedure mentioned above, values of relative amplitude of deflections and moments at the points $n = 0, 1, 2, \ldots, 19$ for the values of $h/a = 0.05$, 0.10, 0.15, $0{\cdot}20$ and the frequencies for the fundamental and first mode of vibrations $p^2a^2/E|\rho = 0.412730$, 0.621038, 0.960809, 1.474358 and 2.067479, 3.00, 4.781607, 7.392046 respectively, satisfying the residuales FR_1 and FR_2 can be obtained. The graphs for the fundamental and first mode of vibration have been drawn for the frequency parameter against h/a (Figs. 2 and 3). For this purpose of numerical work $K/E|\rho$ is taken equal to 0.01.

References

Mindlin, R. D. (1951). Influence of rotatory inertia and shear on flexural motions of isotropic elastic plate. *J. appl. Mech.*, **18**, 31.

Timoshenko, S. P. (1921). On the correction for shear of differential equation for transverse vibrations of prismatical bars. *Phil. Mag.*, **41**, 744.

Sharma, B. B. (1971). Some vibration problems in elasticity. Ph.D. Thesis submitted to Meerut University, p. 81.

Allen, D. N. De G. (1954). Relaxation Methods. McGraw-Hill Book Co., Inc., New York, p. 103.