STIFFENING EFFECT OF CENTRIFUGAL FORCE ON FUNDAMENTAL FREQUENCY OF COUPLED VIBRATION OF A ROTATING SLENDER BEAM UNDER AERODYNAMIC COUPLINGS

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The stiffening effect of centrifugal force on the fundamental frequency and critical speed of coupled vibrations of a rotating slender beam under aerodynamic coupling is investigated by means of a method based on Rayleigh's Quotient. It has been found that the fundamental frequency and critical speed depend upon the ratio of the hub radius to the blade length and upon the angle made by the minor axis of inertia with the direction of the centrifugal velocity.

NOMENCLATURE

U = speed of the steady flow

C = the chord length of the cross section (air foil)

ρ = density of the fluid

α = rotation of the beam cross sections about the elastic axis

x_0 = the distance of elastic axis after the leading edge

x = radial coordinate

y = tangential coordinate

z = axial coordinate

h = deflection of elastic axis in Y direction

e = axis of minor moment of inertia I_{min}.

h' = deflection perpendicular to axis

l = length of the blade in radial direction

m = mass per unit length along the span

E = modulus of elasticity.

I = moment of inertia of the cross section about the axis perpendicular to the plane of bending.

G = shear modulus of rigidity

J = a constant depending upon the cross section of the wing such that GJ is the torsional rigidity of the wing.
δ = distance between the elastic axis and the centroidal axes.
S = area of the cross section of the wing
I_α = mass moment of inertia about the elastic axis per unit length
t = time
M = moment of the centrifugal force at the point
ξ = dimensionless variable
W = frequency of vibration
L = aerodynamic lift about the elastic axis
N = aerodynamic Moment per unit span about the elastic axis.

Introduction

The analysis presented here considers vibration of a slender beam that could represent a turbine blade of simple geometry.

The oscillation of a beam whose elastic axis and the line of centres of gravity do not coincide is always 'coupled'. The beam is attached to a disc of radius, r, as indicated in Fig. 1 and disc rotates with angular velocity. The

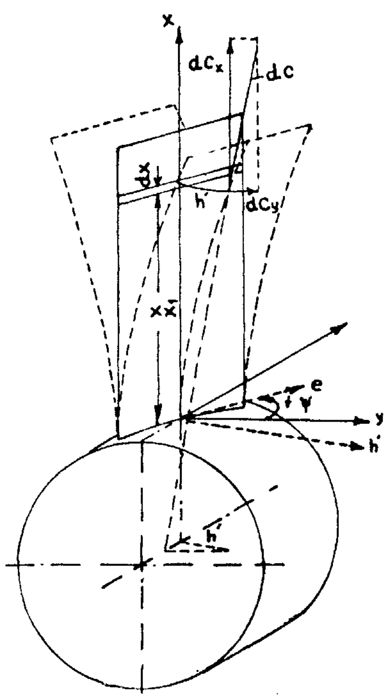

Fig. 1. Cantilever beam attached to a circular disc.

beam is allowed to oscillate in a plane making an angle $\left(\frac{\pi}{2}-\psi\right)$ with the plane of rotation.

The aim of the following investigation is to find out the effect of this angle and the ratio of the hub radius to the blade length on the fundamental frequency and critical speed U of the rotating beam.

For the analysis presented here it is assumed that the elastic axis is a straight line and the deflection of the elastic axis is restrained to the vertical direction only. It is also assumed that S, I and J are constant. This, however is not an essential assumption. If S, I and J are functions of x the amount of computing work is greater than S, I and J independent of x.

The Differential Equations

The differential equations for the deflected form of the neutral axis of a bar according to elementary theory of bending is

$$EI \frac{\partial^4 h}{\partial x^4} = w \qquad (1)$$

where w is the intensity of the distributed load.

If the load is distributed along the centroidal axis the given load can be replaced by the same load distributed along the shear centre axis, and a torque of intensity $w\delta$ distributed along the same axis.

Let the x axis coincide with the shear centre axis. Since the torsion is not uniform the relation between the variable torque T and the angle of attack α is given by (Timoshenko 1955).

$$T = GJ \frac{\partial \alpha}{\partial x} - C_1' \frac{\partial^3 \alpha}{\partial x^3} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots (2)$$

where GJ is the torsional rigidity for uniform torsion and C_1' is the warping rigidity. Differentiating with respect to x, we obtain

$$GJ \frac{\partial^2 \alpha}{\partial x^2} - C_1' \frac{\partial^4 \alpha}{\partial x^4} = \omega\delta$$

with the positive torque as shown in Fig. 2.

For a vibrating bar the intensity of the inertia force is

$$-m \frac{\partial^2}{\partial t^2} (h+\delta\alpha)$$

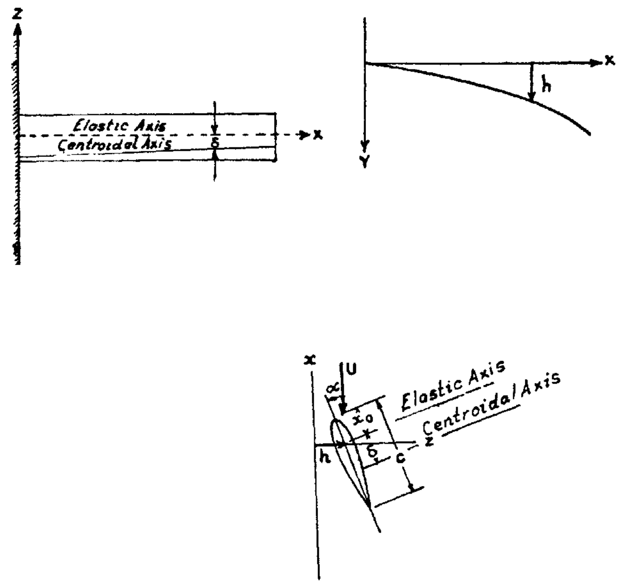

FIG. 2. Deflection and rotation of a cantilever beam from and about elastic axis.

and the intensity of intertia moment about the elastic axis is

$$-I_\alpha \frac{\partial^2 \alpha}{\partial t^2} .$$

The following are the differential equations for the coupled bending and torsional vibrations

$$\left.\begin{aligned} EI \frac{\partial^4 h}{\partial x^4} &= -m \frac{\partial^2 (h+\delta\alpha)}{\partial t^2} \\ GJ \frac{\partial^2 \alpha}{\partial x^2} - C_1' \frac{\partial^4 \alpha}{\partial x^4} &= \delta m \frac{\partial^2}{\partial t^2}(h+\delta\alpha) + I_\alpha \frac{\partial^2 \alpha}{\partial t^2} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (3)$$

when $\delta = 0$, these equations reduce to two independent ones for the purely torsional and purely bending oscillations.

Also in a steady flow of speed U, the blade will have some deformation due to the aerodynamic force. Then the above differential equations become

$$\left.\begin{aligned} & EI \frac{\partial^4 h}{\partial x^4} + m\delta \frac{\partial^2 \alpha}{\partial t^2} + m \frac{\partial^2 h}{\partial t^2} + L = 0 \\ & GJ \frac{\partial^2 \alpha}{\partial x^2} - C_1' \frac{\partial^4 \alpha}{\partial x^4} - m\delta \frac{\partial^2 h}{\partial t^2} - m\left(\delta^2 + \frac{I_\alpha}{m}\right) \frac{\partial^2 \alpha}{\partial t^2} + N = 0 \end{aligned}\right\} \quad \ldots \quad (4)$$

where L and N are given by (Sharma 1971)

$$L = \frac{\rho U^2}{2} c C_L \quad \text{and} \quad N = \frac{\rho U^2}{2} c^2 \left[C_N + \frac{x_0}{c} C_L \right] \qquad \ldots \quad \ldots \quad (5)$$

where C_L and C_N are the lift and moment coefficient about the leading edge and which are expressed as

$$C_L = \frac{dC_L}{d\alpha} \left[\alpha + \frac{1}{U} \frac{dh}{dt} + \frac{1}{U} \left(\frac{3}{4} c - x_0 \right) \frac{d\alpha}{dt} \right] \qquad \ldots \quad \ldots \quad (6)$$

$$C_N = - \frac{c\pi}{\delta U} \frac{d\alpha}{dt} - \frac{1}{4} C_L \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

Refering to Fig. 1 the centrifugal force exerted by a mass element of the radial length $d\eta$ at $x = \eta$ is

$$dc = m\Omega^2(\gamma_0 + \eta) d\eta.$$

If the deflection of the blade at time t is $h(x, t)$, the tangential and radial components dc are

$$dc_Y = dc \ \frac{h' \sin \psi}{\gamma_0 + \eta}$$

and

$$dc_x = dc \left[1 - \left(\frac{h' \sin \psi}{\gamma_0 + \eta} \right)^2 \right]^{\frac{1}{2}}$$

respectively, where ψ is the angle between Y and e axis.

Again let us assume that $\sin \psi << 1$, $dc_x = dc$. The components c_Y and c_x applied at x follow by integrating from $\eta = x$ to $\eta = 1$ and are

$$c_x = m\Omega^2[\gamma_0(l - x) + \tfrac{1}{2}(l^2 - x^2)] \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

$$c_Y = m\Omega^2 \sin \psi \int_x^l h'(x, t) d\eta \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

Refering Fig. 3 the moment dM about the minor principal axis of inertia is

$$dM = c_x dh'(x, t) - c_Y \sin \psi \, dx. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

Equations (8), (9) and (10) with the assumption $h'(x, t) = h(x, t)$.

$$\frac{\partial^2 M}{\partial x^2} = m\Omega^2 \left[\{\gamma_0(l - x) + \tfrac{1}{2}(l^2 - x^2)\} \frac{\partial^2 h}{\partial x^2} - (\gamma_0 + x) \frac{\partial h}{\partial x} + \sin \psi h \right]. \qquad \ldots \quad (11)$$

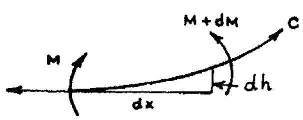

FIG. 3. Showing moment about the minor principal axis.

First equation of (4) is modified by adding the lateral load $-\frac{\partial^2 M}{\partial x^2}$, if the centrifugal force effect is to be considered and then the governing equations become

$$EI\ \frac{\partial^4 h}{\partial x^4}+m\delta\ \frac{\partial^2\alpha}{\partial t^2}+m\ \frac{\partial^2 h}{\partial t^2}+L-\frac{\partial^2 M}{\partial x^2}=0 \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

$$GJ\ \frac{\partial^2\alpha}{\partial x^2}-C_1'\ \frac{\partial^4\alpha}{\partial x^4}-m\delta\ \frac{\partial^2 h}{\partial t^2}-m(\delta^2+I_\alpha/m)\ \frac{\partial^2\alpha}{\partial t^2}+N=0 \quad \ldots \quad (13)$$

where L, N and $\partial^2 M/\partial x^2$ are given by eqns. (5), (6), (7) and (12).

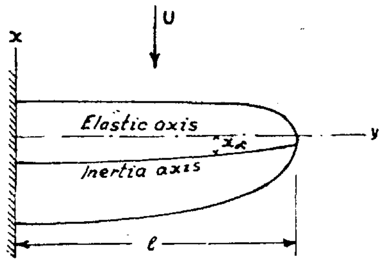

FIG. 4. Notation for a cantilever beam.

DETERMINATION OF NATURAL FREQUENCIES AND CRITICAL SPEEDS

The equations are now put in terms of dimensionless variable $\xi = x/l$ and substitutions

$$\beta=\frac{EI}{ml^4}, \qquad \nu^2=\frac{GJ}{ml^2},$$

$$I_\alpha' = I_\alpha/m, \qquad T = C_1'/ml^4,$$

$$K_1=\frac{\rho c}{2m}\ \frac{dC_L}{d\alpha}, \qquad K_2=\left(\frac{3}{4}-\frac{x_0}{c}\right)c,$$

$$K_3=\frac{\rho c^2}{2m}\ \frac{c\pi}{8}, \qquad K_4=\left(\frac{x_0}{c}-\frac{1}{4}\right)c$$

are used.

The equations (12) and (13) become

$$\beta^2\frac{\partial^4 h}{\partial\xi^4}-\Omega^2\left[\left\{\frac{\gamma_0}{l}+\frac{1}{2}-\frac{\gamma_0}{l}\xi-\frac{1}{2}\xi^2\right\}\frac{\partial^2 h}{\partial\xi^2}-\left(\frac{\gamma_0}{l}+\xi\right)\frac{\partial h}{\partial\xi}+\sin^2\psi h(x,t)\right]$$

$$+\frac{\partial^2 h}{\partial t^2}+\delta\ \frac{\partial^2\alpha}{\partial t^2}+K_1\left(\alpha\, U^2+\frac{U\partial h}{\partial t}+K_2 U\frac{\partial\alpha}{\partial t}\right)=0$$

and

$$\nu^2 \frac{\partial^2\alpha}{\partial\xi^2} - T\frac{\partial^4\alpha}{\partial\xi^4} - (\delta^2+I'_\alpha)\frac{\partial^2\alpha}{\partial t^2} - \delta\frac{\partial^2 h}{\partial t^2} - K_3U\frac{\partial\alpha}{\partial t}$$

$$+K_1K_4\left(U^2\alpha + U\frac{\partial h}{\partial t} + UK_2\frac{\partial\alpha}{\partial t}\right) = 0. \quad \ldots \quad \ldots \quad (14)$$

In the following U represents U_0 and the motion is harmonic representable as follows :

$$h(\xi, t) = Af(\xi)e^{i\omega t}$$

$$\alpha(\xi, t) = B\phi(\xi)e^{i\omega t} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

where ω is real and A and B are complex constants.

These functions $f(\xi)$ and $\phi(\xi)$ satisfy the boundary conditions of the beam which are as follows :

$$h = \frac{\partial h}{\partial\xi} = \frac{\partial^2\alpha}{\partial\xi^2} = \alpha = 0 \qquad \text{at } \xi = 0,$$

$$\frac{\partial^2 h}{\partial\xi^2} = \frac{\partial^3 h}{\partial\xi^3} = \frac{\partial\alpha}{\partial\xi} = \frac{\partial^3\alpha}{\partial\xi^3} = 0 \qquad \text{at } \xi = l. \quad \ldots \quad \ldots \quad (16)$$

For an approximate determination of the fundamental frequency $f(\xi)$ is chosen as the shape function for the fundamental mode of uncoupled bending vibration and $\phi(\xi)$ as the shape function for the fundamental mode of uncoupled torsional vibration in still air, of a cantilever beam of uniform cross-section. The shape function which will statisfy the boundary conditions (16) are

$$\left.\begin{aligned} f(\xi) &= \cosh\lambda\xi - \cos\lambda\xi - 0{\cdot}7341(\sinh\lambda\xi - \sin\lambda\xi) \\ \text{and} \quad \phi(\xi) &= \sin\frac{\pi\xi}{2} \end{aligned}\right\} \quad \ldots \quad (17)$$

where $\lambda = 1{\cdot}87510$.

Substituting eqns. (15) in eqns. (14) we obtain

$$\left.\begin{aligned} &A\left[\beta^2\frac{d^4f}{d\xi^4} - \Omega^2\left\{\frac{\gamma_0}{l} + \frac{1}{2} - \frac{\gamma_0}{l}\xi - \frac{1}{2}\xi^2\right\}\frac{d^2f}{d\xi^2} + \Omega^2\left\{\left(\frac{\gamma_0}{l}+\xi\right)\frac{df}{d\xi} - \sin^2\psi f\right\}\right. \\ &\qquad \left. -\omega^2 f + K_1 i\omega U f\right] + B[-\delta\omega^2\phi + K_1U^2\phi + K_1K_2Ui\omega\phi] = 0 \\ &A[\delta\omega^2 f + K_1K_4Ui\omega f] + B\left[\nu^2\frac{d^2\phi}{d\xi^2} - T\frac{d^4\phi}{d\xi^4} + (\delta^2+I'_\alpha)\omega^2\phi\right. \\ &\qquad \left. + K_1K_4U^2\phi + (K_1K_2K_4 - K_3)Ui\omega\phi\right] = 0. \end{aligned}\right\} (18)$$

For the solution of eqns. (18), we multiply eqns. (18) by f and ϕ respectively and integrating the result with respect to ξ form 0 to 1, we obtain the following equations (Fung 1955) :

$$\left.\begin{aligned} &A(a_1-a_2+a_3-\omega^2a_4+i\omega Ua_5)+B(\omega^2a_6-U^2a_7-i\omega Ua_8) = 0 \\ &A(\omega^2a'_6-i\omega Ua_9)+B(a_{10}-\omega^2a_{11}+U^2a_{12}+iU\omega a_{13}+a_{14}) = 0 \end{aligned}\right\} \quad \ldots (19)$$

where

$$a_1 = \beta^2 \int_0^1 \frac{d^4f}{d\xi^4} fd\xi = \beta^2 \int_0^1 \left(\frac{d^2f}{d\xi^2}\right)^2 d\xi \qquad a_8 = -K_1K_2 \int_0^1 f\phi\alpha\xi$$

$$a_2 = \Omega^2 \int_0^1 \left(\frac{\gamma_0}{l}+\frac{1}{2}-\frac{\gamma_0}{l}\xi-\frac{1}{2}\xi^2\right) \frac{d^2f}{d\xi^2} fd\xi \quad a_9 = K_1K_4 \int_0^1 f\phi d\xi$$

$$a_3 = \Omega^2 \int_0^1 \left\{\left(\frac{\gamma_0}{l}+\xi\right)\frac{df}{d\xi}-f\sin^2\psi\right\} fd\xi \qquad a_{10} = -\nu \int_0^1 \frac{d^2\phi}{d\xi^2}\phi d\xi = \nu^2 \int_0^1 \frac{d\phi^2}{d\xi}d\xi$$

$$a_4 = \int_0^1 f^2d\xi \qquad a_{11} = (\delta^2+I'_\alpha) \int_0^1 \phi^2d\xi$$

$$a_5 = K_1 \int_0^1 f^2d\xi \qquad a_{12} = -K_1K_4 \int_0^1 \phi^2d\xi$$

$$a_6 = a'_6 = -\delta \int_0^1 f\phi d\xi \qquad a_{13} = (K_3-K_1K_2K_4) \int_0^1 \phi^2d\xi$$

$$a_7 = -K_1 \int_0^1 f\phi d\xi \qquad a_{14} = T \int_0^1 \frac{d^4\phi}{d\xi^4}\phi d\xi.$$

This determinant being complex, both the real and imaginary parts must vanish. On setting the determinant to zero and separating the real and imaginary parts, we obtain the following equations.

$$\left.\begin{aligned} A_1\omega^4-(C_1+C_2U^2)W^2+(E_1+E_2U^2) &= 0 \\ -B_1\omega^2+(B_2+B_3U^2) \qquad &= 0 \end{aligned}\right\} \quad \ldots \quad \ldots \quad (20)$$

where

$$A_1 = a_4a_{11}-a_6a'_6, \qquad E_1 = (a_{10}+a_{14})(a_1-a_2+a_3),$$
$$C_1 = a_4(a_{10}+a_{14})+a_{11}(a_1-a_2+a_3), \qquad E_2 = a_{12}(a_1-a_2+a_3).$$
$$C_2 = a_4a_{12}+a_5a_{13}-a'_6a_7-a_8a_9,$$

The second equation of (20) gives us

$$\omega^2 = \frac{B_2+B_3U^2}{B_1} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (21)$$

Substituting this expression into the first equation of (20) we obtain

$$PU^4-QU^2+R = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (22)$$

where

$$P = B_3(B_1C_2-A_1B_3),$$

$$Q = 2B_2B_3A_1-C_1B_1B_3-C_2B_1B_2+B_1^2E_2,$$

$$R = C_1B_1B_2-A_1B_2^2-B_1^2E_1.$$

From eqn. (22) we obtain the critical speed

$$U^2 = \frac{Q\pm\sqrt{Q^2-4PR}}{2P}. \quad \ldots \quad \ldots \quad \ldots \quad (23)$$

The right-hand side of eqn. (23) is positive. Corresponding to the two solutions of U^2 from eqn. (23). there are two values of ω^2 from equation (21). Usually the smaller U^2 is associated with the higher ω^2 in eqn. (21). The coefficients B_1 and B_2 are always positive where as B_3 is negative if the elastic axis lies behind the 1/4-chord point, as is usually so.

The larger of the two values of U^2 given by eqn. (23) will provide the smaller value of ω^2 which will be the upper bound for the frequency of the fundamental mode of vibration. The smaller of the two values of U^2 will provide the larger value of ω^2 which will be an upper bound for the next higher mode of vibration.

Numerical Example

A numerical example for the coupled torsional vibrations of a slender rotating beam involving elastic and aerodynamic forces just described are presented here. Critical speeds and frequencies are computed from eqns. (23) and (21) and the cross-section of the blade is taken as semi-circle of radius γ_0 and thickness t_1. The physical constants of the blade and other constants are given below (Biezeno and Grammel 1954) :

$S = 0{\cdot}14889$ in^2

$a_0 = c = 0{\cdot}24885$ in.

$t_1 = 0{\cdot}19712$ in.

$m = 0{\cdot}00011$ lbs.

$\Omega = 314$ sec^{-1}

$$C_1' = 0{\cdot}000189\times 10^6 = Ea_0^5t_1\left(\frac{\pi^3}{12}-\frac{8}{\pi}\right)$$

$I_\alpha = 0{\cdot}095235$ m lbs/in^2

$$= m\left\{a_0^2+\left(\frac{2a_0}{\pi}+0{\cdot}024\right)^2\right\}$$

$$\delta = 0{\cdot}024$$

$$\frac{dC_L}{d\alpha} = 6 \text{ radians} < 2\pi$$

$$J = 0{\cdot}00193 \text{ in}^4 = \frac{St_1^3}{3}$$

$$E = 29{\cdot}20 \times 10^6 \text{ lbs/in}^2$$

$$I = 0{\cdot}001845 \text{ in}^4$$

$$G = 11{\cdot}53 \times 10^6 \text{ lbs/in}^2$$

$$\rho = 4{\cdot}67161 \times 10^{-5} \text{ lbs/in}^3.$$

With these values, the results for the ratio $\gamma_0/l = 1, 2, 3, 4$ and for positions of $\psi = 0°, 90°$ when $\delta \neq 0$ and $\delta = 0$ can be found out. The respective frequencies are the smaller of the two values of ω^2 calculated from (21).

The respective speed of the steady flow are the larger of the two values of U^2 calculated from (23) i.e., the critical speed corresponding to torsional coupled vibration of the rotating slender beam under aerodynamic couplings. The smaller of the two values of U^2 will provide the frequency for the next higher mode of vibration. It can also be seen from computer the numerical valuees of U^2 and ω^2 for various ratio γ_0/l that as U^2 decreases the corresponding value of ω^2 increases.

Concluding Remarks

The results thus obtained show that for a system under consideration the lowest frequencies and the highest critical speed depend upon the ratio of the hub radius to the blade length, and upon setting the angle ψ made by the blade chord and the rotational velocity. The change in the frequency and speed parameter is a linear function of the hub radius.

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