PERIODIC ORBITS OF COLLISION IN THE RESTRICTED PROBLEM OF THREE BODIES, WHEN THE BIGGER PRIMARY IS AN OBLATE SPHEROID

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In this paper the problem is regularized in the neighbourhood of bigger finite mass, which has been taken as oblate spheroid and the existence of periodic orbits, symmetric as well as doubly symmetric has been studied. Lastly, it is seen that periodic orbits of collision also exist.

1. INTRODUCTION

Giacaglia (1967) studied the motion of an infinitesimal mass under the gravitational field of two finite masses. He showed the existence of periodic collision orbits for the infinitesimal mass when the finite masses were moving in circular orbits and they were taken point-masses.

In this paper we have taken the bigger primary to be an oblate spheroid. The equatorial plane of the spheroid will be taken to be coincident with the plane of the circular orbits.

We shall also study the existence of periodic orbits, including the case of collision.

2. CANONICAL ELEMENTS FOR THE REGULARIZED RESTRICTED PROBLEM

In canonical form the equations of motion are

$$x_i = \frac{\partial H}{\partial p_i}, \quad p_i = -\frac{\partial H}{\partial x_i} \qquad (i = 1, 2) \qquad \ldots \quad \ldots \quad (1)$$

where

$$H = \tfrac{1}{2}(p_1^2+p_2^2)+(x_2p_1-x_1p_2)-\frac{(1-\mu)}{r_1}-\frac{\mu}{r_2}-\frac{I}{2r_1^3} = C = \text{const.}$$

and

$$C = C(\mu) = C(0)+\mu C_1(\mu) = C_0+\mu C_1, \; p_1 = \dot{x}_1-x_2, \; p_2 = \dot{x}_2+x_1.$$

The x_i are synodic rectangular dimensionless coordinates as defined by Szebehely and Giacaglia (1964).

$I = (I_3 - I_e) > 0$ is a constant quantity, where I_3 is axial moment of inertia and I_e is equatorial moment of inertia of the oblate spheroid.

The term $-I/2r_1^3$ comes due to the oblateness (McCuskey 1963, p. 166) of the bigger primary.

For the regularization of the solutio ns, we introduce Levi-Civita's (1906) parabolic transformation

$$S = (\mu + \xi_1^2 - \xi_2^2)p_1 + 2\xi_1\xi_2 p_2$$

and such that

$$x_i = \frac{\partial S}{\partial p_i}, \quad \pi_i = \frac{\partial S}{\partial \xi_i} \quad (i = 1, 2) \qquad \ldots \qquad \ldots \quad (2)$$

Here π_i are the momenta associated with the new coordinates.

The Hamiltonian given in terms of these new variables is

$$H = \frac{\pi^2}{8r_1} + \tfrac{1}{2}(\xi_2\pi_1 - \xi_1\pi_2) - \frac{\mu}{2r_1}(\xi_1\pi_2 + \xi_2\pi_1) - \frac{\mu}{r_2} - \frac{(1-\mu)}{r_1} - \frac{I}{2r_1^3} = C$$

where

$$r_1 = \xi_1^2 + \xi_2^2 = \xi^2, \quad r_2^2 = 1 + 2(\xi_1^2 - \xi_2^2) + \xi^4, \quad \pi^2 = \pi_1^2 + \pi_2^2.$$

We now introduce a new independent variable τ, defined by

$$dt = r_1 d\tau, \quad t = 0 \quad \text{at } \tau = 0.$$

The equations of motion become

$$\xi_i' = \frac{\partial K}{\partial \pi_i}, \quad \pi_i' = -\frac{\partial K}{\partial \xi_i} \qquad (i = 1, 2) \qquad \ldots \qquad \ldots \quad (3)$$

where primes denote derivatives with respect to τ and

$$K = r_1(H - C) = \frac{1}{8}\pi^2 + \tfrac{1}{2}\xi^2(\xi_2\pi_1 - \xi_1\pi_2 - 2C) - \\ - \tfrac{1}{2}\mu(\xi_1\pi_2 + \xi_2\pi_1 + 2r_1/r_2) - (1-\mu) - \frac{I}{2r_1^2}. \qquad \ldots \quad (4)$$

The quantity K given by eqn. (4) is zero along any solution of the differential equations (3) and *vice versa* every solution of these equations must belong to the $K = 0$ manifold. In other words, the first integral $K = \text{const.}$ obtained from eqns. (3) has a value which is not arbitrary. It must be zero.

The Hamiltonian K can be written as

$$K = K_0 + K_1 + \mu K_2$$

where

$$\left.\begin{aligned} K_0 &= \frac{1}{8}\pi^2+\tfrac{1}{2}\xi^2(\xi_2\pi_1-\xi_1\pi_2-2C_0)-1 \\ K_1 &= -I/2r_1^2, \\ K_2 &= -\tfrac{1}{2}(\xi_1\pi_2+\xi_2\pi_1+2r_1/r_2)-\left[\frac{C-C_0}{\mu}\right]\xi^2+1. \end{aligned}\right\} \quad \ldots \quad (5)$$

Now K_0 has the same expression as in Giacaglia (1967) and with him we shall assume that K_0 is negative (i.e., $K_0 = -\epsilon < 0$, $|\epsilon| < 1$).

The solution corresponding to K_0 is given by Giacaglia (1967).

The Hamiltonian K_1 in terms of the canonical elements (l, L, g, G) defined by Giacaglia (1967) is written as

$$K_1 = \frac{I(G+C_0)}{[L^2+(L^2-G^2)\cos^2 l-2L(L^2-G^2)^{\frac{1}{2}}\cos l]} \quad \ldots \quad (6)$$

Hamilton's equations of motion for the Hamiltonian K_1 are

$$\frac{d\alpha_r}{d\tau} = \frac{\partial K_1}{\partial \beta_r}, \quad \frac{d\beta_r}{d\tau} = -\frac{\partial K_1}{\partial \alpha_r} \quad (r = 1, 2) \quad \ldots \quad \ldots \quad (7)$$

where $\alpha_1 = l$, $\alpha_2 = g$; $\beta_1 = L$, $\beta_2 = G$.

From (6) and (7), we have

$$\frac{dl}{d\tau} = \frac{\partial K_1}{\partial L} = \frac{2I(G+C_0)[(L^2-G^2)^{1/2}\cos^2 l+L^2\cos l(L^2-G^2)^{-1/2}-L(1+\cos^2 l)]}{[L^2+(L^2-G^2)\cos^2 l-2L(L^2-G^2)^{1/2}\cos l]^2} \quad \ldots \quad (8)$$

$$\frac{dg}{d\tau} = \frac{\partial K_1}{\partial G} = \frac{I}{[L^2+(L^2-G^2)\cos^2 l-2L(L^2-G^2)^{1/2}\cos l]} + \frac{2IG(G+C_0)[\cos^2 l-L(L^2-G^2)^{-1/2}\cos l]}{[L^2+(L^2-G^2)\cos^2 l-2L(L^2-G^2)^{1/2}\cos l]^2} \quad \ldots \quad (9)$$

$$\frac{dL}{d\tau} = -\frac{\partial K_1}{\partial l} = \frac{-I(G+C_0)(L^2-G^2)[\sin 2l-2L(L^2-G^2)^{-1/2}\sin l]}{[L^2+(L^2-G^2)\cos^2 l-2L(L^2-G^2)^{1/2}\cos l]^2} \quad \ldots \quad (10)$$

$$\frac{\partial G}{d\tau} = -\frac{\partial K_1}{\partial g} = 0. \quad \ldots \quad \ldots \quad (11)$$

Solution corresponding to K_0 gives ξ_1, ξ_2, π_1, π_2 in terms of the canonical elements l, L, g, G. Eqns. (8), (9) and (10) define l, g, L as function of time, which on substitution in the solution corresponding to K_0 give us the values of ξ_1, ξ_2, π_1, π_2 as functions of canonical elements and time. This will give us the solution corresponding to the Hamiltonian (K_0+K_1).

Now the complete Hamiltonian is

$$K = K_0+K_1+\mu K_2.$$

Hamilton's equations of motion for the Hamiltonian K are

$$\left.\begin{aligned}
\frac{dl}{d\tau} &= \frac{\partial K}{\partial L} = \frac{\partial K_0}{\partial L}+\frac{\partial K_1}{\partial L}+\mu\left(\frac{\partial K_2}{\partial L}\right)\\
\frac{dg}{d\tau} &= \frac{\partial K}{\partial G} = \frac{\partial K_0}{\partial G}+\frac{\partial K_1}{\partial G}+\mu\left(\frac{\partial K_2}{\partial G}\right)\\
\frac{dL}{d\tau} &= -\frac{\partial K}{\partial l} = -\frac{\partial K_0}{\partial l}-\frac{\partial K_1}{\partial l}-\mu\left(\frac{\partial K_2}{\partial l}\right)\\
\frac{dG}{d\tau} &= -\frac{\partial K}{\partial g} = -\frac{\partial K_0}{\partial g}-\frac{\partial K_1}{\partial g}-\mu\left(\frac{\partial K_2}{\partial g}\right)
\end{aligned}\right\} \quad \ldots \quad \ldots \quad (12)$$

where

$$K_2 = -\{eL\sin l\sin 2\varphi+G\cos 2\varphi+a(1-e\cos l)$$
$$\times[1+2a(1-e\cos l)\cos 2\varphi+a^2(1-e\cos l)^2]^{-\frac{1}{2}}\}-C_1a(1-e\cos l). \quad \ldots \quad (13)$$

Equations (12) form the basis of general perturbation theory for the problem in question.

3. Existence of Periodic Orbits when $\mu \neq 0$

Here we shall follow the method used by Choudhry (1966) and Bhatnagar (1969) to prove the existence of periodic orbits when $\mu \neq 0$.

The Hamiltonian K can be written as

$$K = (K_0+K_1)+\mu K_2 = R_0+\mu R_1$$

where

$$R_0 = \frac{1}{8}\pi^2+\tfrac{1}{2}\xi^2(\xi_2\pi_1-\xi_1\pi_2-2C_0)-1-\frac{I}{2r_1^2}$$
$$= L[-2(G+C_0)]^{\frac{1}{2}}-1-I/2r_1^2$$

and

$$R_1 = -\tfrac{1}{2}(\xi_1\pi_2+\xi_2\pi_1+2r_1/r_2)-\left[\frac{(C-C_0)}{\mu}\right]\xi^2+1$$

where

$$\xi_1\pi_2+\xi_2\pi_1 = 2eL\sin l\sin 2\phi+2G\cos 2\phi$$
$$r_1 = \xi^2 = a(1-e\cos l)$$
$$r_2^2 = 1+2a(1-e\cos l)\cos 2\phi+a^2(1-e\cos l)^2$$
$$(C-C_0) = \mu C_1.$$

The orbits will be periodic for $\mu \neq 0$ if the following conditions (Duboshn 1964) are satisfied :

$$\frac{\partial[R_1]}{\partial \omega_i} = 0 \qquad (i = 1, 2) \qquad \ldots \quad \ldots \quad (14)$$

$$\frac{\partial[R_1]}{\partial a_i} = 0 \qquad (i = 1, 2) \qquad \ldots \quad \ldots \quad (15)$$

$$\frac{\partial(\xi_2, \eta_1, \eta_2)}{\partial(\gamma_2, \beta_1, \beta_2)} \neq 0 \quad \text{when } \mu = \beta_i = \gamma_i = 0 \qquad \ldots \quad (16)$$

The general solution in the neighbourhood of the generating solution is given by

$$\left.\begin{aligned} x_i &= a_i + \beta_i + \xi_i(\zeta) \\ y_i &= n_i^{(0)}\zeta + \omega_i + \gamma_i + \eta_i(\zeta) \qquad (i = 1, 2) \end{aligned}\right\} \qquad \ldots \quad \ldots \quad (17)$$

where

$$x_1 = L, \quad x_2 = G; \quad y_1 = l, \quad y_2 = g.$$

(16) can be written (Bhatnagar 1969) as

$$\begin{vmatrix} \dfrac{\partial^2(R_1)}{\partial \omega_2^2} & 0 & 0 \\ 0 & \tau_0 \dfrac{\partial^2 R_0}{\partial a_1^2} & \tau_0 \dfrac{\partial^2 R_0}{\partial a_1 \partial a_2} \\ 0 & \tau_0 \dfrac{\partial^2 R_0}{\partial a_1 \partial a_2} & \tau_0 \dfrac{\partial^2 R_0}{\partial a_2^2} \end{vmatrix} \neq 0 \qquad \ldots \quad (18)$$

Taking zero order terms, we have

$$\left.\begin{aligned} r_1 &= a \\ r_2^2 &= 1 + 2a\cos 2\phi + a^2 \\ \xi_1\pi_2 + \xi_2\pi_1 &= 2G\cos 2\phi \\ R_0 &= L[-2(G+C_0)]^{\frac{1}{2}} - 1 - I/2a^2 \\ R_1 &= -G\cos 2\phi - \frac{a}{r_2} - C_1 a. \end{aligned}\right\} \qquad \ldots \quad \ldots \quad (19)$$

Putting the value of 'a' in terms of the canonical elements in the expression for R_0, we have

$$\begin{aligned} R_0 &= L[-2(G+C_0)]^{\frac{1}{2}} - 1 + \frac{(G+C_0)I}{L^2} \\ &= x_1[-2(x_2+C_0)]^{\frac{1}{2}} - 1 + \frac{(x_2+C_0)I}{x_1^2} \\ &= a_1[-2(a_2+C_0)]^{\frac{1}{2}} - 1 + \frac{(a_2+C_0)I}{a_1^2} \end{aligned}$$

Therefore,

$$\frac{\partial^2 R_0}{\partial a_1^2} = \frac{6(a_2+C_0)I}{a_1^4}, \quad \frac{\partial^2 R_0}{\partial a_2^2} = \frac{-a_1}{[-2(a_2+C_0)]^{3/2}}$$

$$\frac{\partial^2 R_0}{\partial a_1 \partial a_2} = \frac{-1}{[-2(a_2+C_0)]^{\frac{1}{2}}} - \frac{2I}{a_1^3}.$$

The value of the determinant (18) becomes

$$= \frac{\partial^2[R_1]}{\partial \omega_2^2}\left[\tau_0^2\left\{-\left(\frac{1}{n_l^2}+\frac{4I^2}{a_1^6}+\frac{I}{n_l a_1^3}\right)\right\}\right] \qquad \ldots \quad \ldots (20)$$

The quantity in the brackets is not equal to zero Because $I > 0$

$$n_l = [-2(a_2+C_0)]^{\frac{1}{2}} = [-2(G+C_0)]^{\frac{1}{2}} > 0$$

$$a_1 = L = \frac{\alpha}{[-2(G+C_0)]^{\frac{1}{2}}} > 0 \quad \text{(Giacaglia 1967).}$$

Since the quantity in the brackets in (20) is not equal to zero, hence (16) will not be equal to zero if

$$\frac{\partial^2[R_1]}{\partial \omega_2^2} \neq 0.$$

From (19), for the zero order terms, we have

$$R_1 = -G\cos 2\phi - \frac{a}{r_2} - C_1 a$$

where

$$r_2^2 = 1+2a\cos 2\phi + a^2$$

$$\phi = \tfrac{1}{2}(y_1+y_2) - \frac{(x_1^2-x_2^2)^{\frac{1}{2}}\sin y_1}{4(x_2+C_0)}$$

$$x_i = a_i, \quad y_i = n_i^{(0)}\zeta + \omega_i \qquad (i = 1, 2).$$

Condition (14) gives

$$\frac{\partial[R_1]}{\partial \omega_2} = \sin 2\phi\left[G - \frac{a^2}{r_2^3}\right] = 0$$

i.e. either $2\phi = 0, \pi$ etc. or $G = \dfrac{a^2}{r_2^3}$.

Now

$$\frac{\partial^2[R_1]}{\partial \omega_2^2} = \cos 2\phi\left(G - \frac{a^2}{r_2^3}\right) - \frac{3a^2\sin^2 2\phi}{r_2^5}$$

when

$$2\phi = 0,\ \pi \text{ etc. or } \quad G = \frac{a^2}{r_2^3}$$

$$\frac{\partial^2[R_1]}{\partial \omega_2^2} \neq 0.$$

We may also note that

$$\frac{\partial[R_1]}{\partial \omega_1} = \sin 2\phi \left(G - \frac{a^2}{r_2^3} \right) \left[1 - \frac{(x_1^2 - x_2^2)^{\frac{1}{2}}}{2(x_2 + C_0)} \cos y_1 \right] = 0$$

when

$$2\phi = 0,\ \pi \text{ etc.} \quad \text{or} \quad G = \frac{a^2}{r_2^3}.$$

Therefore, the conditions (14) are satisfied.

Further

$$\frac{\partial[R_1]}{\partial a_1} = -\sin 2\phi \left\{ G - \frac{a^2}{r_2^3} \right\} \left\{ \frac{x_1(x_1^2 - x_2^2)^{-\frac{1}{2}}}{2(x_2 + C_0)} \right\} \sin y_1 \frac{\partial x_1}{\partial a_1} = 0$$

$$\frac{\partial[R_1]}{\partial a_2} = \mathrm{Sin}\, 2\phi \left\{ G - \frac{a^2}{r_2^3} \right\} \left\{ \frac{x_2(x_2 + C_0)(x_1^2 - x_2^2)^{-\frac{1}{2}} + (x_1^2 - x_2^2)}{2(x_2 + C_0)^2} \right\} \times \sin y_1 \frac{\partial x_2}{\partial a_2} = 0$$

when

$$2\phi = 0,\ \pi \text{ etc.} \quad \text{or} \quad G = \frac{a^2}{r_2^3}.$$

Hence all the conditions, viz. (14), (15) and (16) for the existence of periodic orbits when $\mu \neq 0$ are satisfied.

4. Symmetric or Doubly Symmetric Periodic Orbits when $\mu \neq 0$

The equations of motion (3) in the form

$$\xi_i'' = f_i(\xi, \xi') \qquad (i = 1, 2)$$

are invariant under the transformations

(1) $\xi_1 \to -\xi_1, \quad \xi_2 \to \xi_2, \quad \tau \to -\tau$

(2) $\xi_1 \to \xi_1, \quad \xi_2 \to -\xi_2, \quad \tau \to -\tau$

which shows that symmetric or doubly symmetric orbits may exist for any value of μ.

It may be further noted that when $\xi_2 = 0$ and $\pi_1 = 0$, we have

$$\xi_1' = \frac{\partial K}{\partial \pi_1} = \frac{\pi_1}{4} + \tfrac{1}{2}\xi^2\xi_2 - \tfrac{1}{2}\mu\xi_2 = 0.$$

and when $\xi_1 = 0$ and $\pi_2 = 0$, we have

$$\xi_2' = \frac{\partial K}{\partial \pi_2} = \frac{\pi_2}{4} - \tfrac{1}{2}\xi^2\xi_1 - \tfrac{1}{2}\mu\xi_1 = 0.$$

Thus we may conclude the following :

(a) If at $\tau = 0$, $\xi_2(0) = 0$, $\pi_1(0) = 0$.

$[\xi_1(0) = 0$ and $\pi_2(0) = 0]$, and at $\tau = \tau_1$,

$\xi_2(\tau_1) = 0$, $\pi_1(\tau_1) = 0$, $[\xi_1(\tau_1) = 0$, $\pi_2(\tau_1) = 0]$

then the orbit is periodic and the period is $2\tau_1$.

(b) If at $\tau = 0$, $\xi_2(0) = 0$, $\pi_1(0) = 0$ and at $\tau = \tau_2$, $\xi_1(\tau_2) = 0$ and $\pi_2(\tau_2) = 0$ then the orbit is periodic and the period is $4\tau_2$.

Case (a) corresponds to orbits symmetric with respect to ξ_1—axis (ξ_2—axis) and case (b) corresponds to orbits symmetric with respect to both the axes (doubly symmetric).

5. Periodic Orbits of Collision when $\mu \neq 0$

Now we shall prove the existence of periodic orbits of collision when $\mu \neq 0$.

Levi-Civita (1904) has proved a relation which is satisfied along a collision orbit in the restricted orbit. He showed that the invariant relation for collision orbits can be analytically continued from the one that corresponds to the problem of two bodies. In our case it is given by $G = 0$. Therefore, when $\mu \neq 0$, this will change into

$$G + \mu F(l, L, g, G, \mu) = 0 \qquad \ldots \qquad \ldots (21)$$

for μ sufficiently small.

Levi-Civita's condition for collision is

$$\theta + 1 = \rho f(\rho, \theta) \qquad \ldots \qquad \ldots (22)$$

where

$$\tan\theta = \frac{x_2}{(x_1 - \mu)}, \quad \rho = r_1^{1/2}.$$

Transforming to the canonical elements previously defined (Bhatnagar 1969), we have

$$\theta = 2\phi, \quad \phi' = \frac{d\phi}{d\tau} = \frac{G}{2r_1} - \frac{r_1}{2}.$$

Condition (22) becomes

$$G - \xi^5 \dot{f}(\xi, 2\phi) = 0 \qquad \ldots \quad \ldots \quad (23)$$

which corresponds to eqn. (21).

At $\tau = 0$, eqn. (23) is automatically satisfied, since there both $G(0)$ and $\xi(0)$ are zero. Moreover, when $\xi = 0$, it follows that $G = 0$. Hence the proof of the existence of periodic orbits in the collision is established.

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