BRA AND KET VECTORS IN INFINITESIMAL GEOMETRY

by N. N. Ghosh, *Indian Association for the Cultivation of Science, Calcutta* 32

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Integral transformation formulae representing 'bra' and 'ket' vectors in infinitesimal geometry have been framed. Associated with the second-order infinitesimal vector operator of this transformation more general bra and ket vectors have been defined noting the linear dependence among the different classes of vectors.

1. Referred to the functional basis of the infinitesimal geometry (Ghosh 1970) let the components of an infinitesimal vector of the first order $f = f(x)\sqrt{dx}$ be denoted by

$$f(x_q)\sqrt{dx_q} \qquad q = 1, 2, \ldots, \infty. \qquad \ldots \quad (1.1)$$

Let $\phi = \phi(x)\sqrt{dx}$ denote another infinitesimal vector having components

$$\phi(x_p)\sqrt{dx_p} \qquad p = 1, 2, \ldots, \infty. \qquad \ldots \quad (1.2)$$

A transition from f to ϕ is then represented by the integral transformation formula

$$\phi(x_p) = \sum_{q=1}^{\infty} \mu\,(x_p x_q) f(x_q) dx_q. \qquad \ldots \quad \ldots \quad (1.3)$$

Analysing the above we notice the existence of an infinitesimal vector of the second order with components

$$\mu(x_p x_q)\sqrt{dx_p}\,\sqrt{dx_q} \qquad \ldots \quad \ldots \quad (1.4)$$

behaving as an infinite matrix operator in (1.3). An alternative form of expressing (1.3) is given by

$$\phi(x_p) = \sum_{q=1}^{\infty} f(x_q)\mu(x_q x_p) dx_q. \qquad \ldots \quad \ldots \quad (1.5)$$

We now introduce the unit infinitesimal vector operator of the second order

$$U = U(x_r x_s)\sqrt{dx_r}\,\sqrt{dx_s} \qquad r, s = 1, 2, \ldots, \infty \qquad \ldots \quad (1.6)$$

transforming *a* vector f into itself.

From the transformation formula

$$f(x_s)\sqrt{dx_s} = \left[\sum_{r=1}^{\infty} U(x_s x_r) f(x_r) dx_r\right]\sqrt{dx_s} \qquad \dots \qquad \dots \quad (1.7)$$

we can postulate in (1.6)

$$\left.\begin{aligned} \sum_{r=1}^{\infty} U(x_s x_r) dx_r &= 1 \qquad r = s \\ U(x_r x_s) &= 0 \qquad r \neq s. \end{aligned}\right\} \qquad \dots \quad (1.8)$$

With the infinitesimal vector U as defined above and in view of the transformation formulae (1.3) and (1.5) we now introduce two types of infinitesimal vectors of the first order expressed as

(i) $$\sum_{q=1}^{\infty} U(x_r x_q) f(x_q) dx_q \sqrt{dx_r}. \qquad \dots \qquad \dots \quad (1.9)$$

(ii) $$\sum_{q=1}^{\infty} f(x_q) U(x_q x_r) dx_q \sqrt{dx_r}. \qquad \dots \qquad \dots \quad (1.10)$$

The former, a column-vector we call 'ket' f and the latter a row-vector, we call 'bra' f. In short form we write (1.3) and (1.5) as

$$\left.\begin{aligned} U\phi &= \mu f \\ \phi U &= f\tilde{\mu} \end{aligned}\right\}. \qquad \dots \qquad \dots \quad (1.11)$$

The ket and bra vectors as defined above undergo two kinds of product

$$fUg \quad \text{and} \quad UfgU \qquad \dots \qquad \dots \quad (1.12)$$

of which the first signifies $\int_a^b f(x)g(x)dx$ while the second is an infinitesimal vector of the second order whose (p, q)th element is of the form

$$f(x_p)\phi(x_q)\sqrt{dx_p}\sqrt{dx_q} \qquad \dots \qquad \dots \quad (1.13)$$

2. Associated with the general infinitesimal vector of the second order

$$\mu(x_r x_s)\sqrt{dx_r}\sqrt{dx_s} \qquad r, s = 1, 2, \dots, \infty \qquad \dots \quad (2.1)$$

in (1.3, 5) there are infinite number of basic ket vectors

$$\sum_{r=1}^{\infty} U(x_p x_r)\mu(x_r x_h) dx_r \sqrt{dx_p} \qquad h = 1, 2, \dots, \infty \qquad \dots \quad (2.2)$$

and an infinite number of basic bra vectors

$$\sum_{r=1}^{\infty} \mu(x_h x_r) U(x_r x_q) dx_r \sqrt{dx_q} \qquad h = 1, 2, \dots, \infty. \qquad \dots \quad (2.3)$$

Using the symbol μ_r^s to denote $\mu(x_r x_s)$ a functional element of the matrix involved in (2.1) let us consider a minor determinant of sth order expressed as

$$\mu^{ij\ldots n}_{ab\ldots f} \qquad \ldots \qquad \ldots \qquad (2.4)$$

with the principal diagonal elements μ_a^i, μ_b^j, ... A ket vector of the sth class is then defined by

$$\sum_{r=1}^{\infty} U(x_p x_r)\mu^{ij\ldots n}_{ab\ldots r}\, dx_r \sqrt{dx_p} \qquad \ldots \qquad \ldots \qquad (2.5)$$

which is shortly written as $\mu^{ij\ldots n}_{ab\ldots p}$, the infinitesimal units $\sqrt{dx_p}$ being implicit in the notation. A bra vector of the sth class is defined by

$$\sum_{s=1}^{\infty} \mu^{ij\ldots s}_{ab\ldots f}\; U(x_s x_q) dx_s \sqrt{dx_q} \qquad \ldots \qquad \ldots \qquad (2.6)$$

written simply as $\qquad M^{ijk\ldots q}_{ab\ldots f}$

According to this notation the basic ket and bra vectors are represented respectively by M_p^h, M_h^q. Continuing similar extension of (2.1) we define an infinitesimal vector of the second order of the sth class by

$$M^{ij\ldots nq}_{ab\ldots fp} = \frac{\mu^{ij\ldots nq}_{ab\ldots fp}}{\mu^{ij\ldots n}_{ab\ldots f}} \sqrt{dx_p}\;\; \sqrt{dx_q}. \qquad \ldots \qquad \ldots \qquad (2.7)$$

Hence (2.1) will have the class no zero.

Referring to a previous paper (Ghosh 1945) we now write down some typical formulae showing the dependence of ket and bra vectors of higher classes to those of lower classes.

$$\left.\begin{aligned}
M^{ijkl}_{abcp} &= \mu^{ijk}_{abc}M^{l}_{p} - \mu^{ijl}_{abc}M^{k}_{p} - \mu^{ilk}_{abc}M^{j}_{p} - \mu^{ljk}_{alc}M^{i}_{p},\\
\mu^{i}_{a}M^{ijkl}_{abcp} &= \mu^{ijk}_{abc}M^{il}_{ap} - \mu^{ijl}_{xbc}M^{ik}_{ap} - \mu^{ilk}_{abc}M^{ij}_{ap},\\
\mu^{ij}_{ab}M^{ijkl}_{abcp} &= \mu^{ijk}_{abc}M^{ijl}_{abp} - \mu^{ijl}_{abc}M^{ijk}_{abp}.\\
M^{ijkq}_{abcd} &= \mu^{ijk}_{abc}M^{q}_{d} - \mu^{ijk}_{abd}M^{q}_{c} - \mu^{ijk}_{cdc}M^{q}_{b} - \mu^{ijk}_{dbc}M^{q}_{a},\\
\mu^{i}_{a}M^{ijkq}_{abcd} &= \mu^{ijk}_{abc}M^{iq}_{ad} - \mu^{ijk}_{abd}M^{iq}_{ac} - \mu^{ijk}_{adc}M^{iq}_{ab},\\
\mu^{ij}_{ab}M^{ijkq}_{abcd} &= \mu^{ijk}_{abc}M^{ijq}_{abd} - \mu^{ijk}_{abd}M^{ijq}_{abc}.
\end{aligned}\right\} \ldots \qquad (2.8)$$

The general rule of writing down such formulae is obvious. The following relations show that the difference of a pair of infinitesimal vectors of the second order belonging to any two classes is expressible as the sum of a number of products of ket and bra vectors, Some typical formulae are given below

$$\mu^{ijk}_{abc}(M^{q}_{p}-M^{ijkq}_{abcp}) = M^{k}_{p}M^{ijq}_{abc}+M^{j}_{p}M^{ikq}_{acb}+M^{i}_{p}M^{jkq}_{bca},$$

$$\mu^{ijk}_{abc}, \mu^{i}_{a}(M^{iq}_{ap}-M^{ijkq}_{abcp}) = M^{ik}_{ap}M^{ijq}_{abc}+M^{ij}_{ap}M^{ikq}_{acb}, \qquad \ldots \quad (2.9)$$

$$\mu^{ijk}_{abc}\mu^{ij}_{ab}(M^{ijq}_{abp}-M^{ijkq}_{abcp}) = M^{ijk}_{abp}M^{ijq}_{abc}.$$

A class of identities involving the product of a determinant and any one of its minors expressed as an aggregate of products of pairs of minors is associated in the proof of the above formulae (Ghosh 1945).

References

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