HEAT TRANSFER EFFECTS FOR AN ELASTICO-VISCOUS FLUID IN A CURVED PIPE

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Heat transfer for hydrodynamically and thermally fully developed laminar flow in a curved pipe is solved by a method of series expansion for a visco-elastic fluid. The curvature of the pipe is assumed to be small. The expressions determining the temperature distribution in the pipe are evaluated to a first order of approximation and temperature profiles are shown in the cross-section of the pipe. The Nusselt number has been calculated by considering second order approximations for the temperature distribution. The case of Newtonian fluid is deduced from this analysis which shows good agreement with experimental observations. It is found that the value of the parameter $L = 2(R_e)^2 a/R$ (R_e is the Reynolds number, a radius of the pipe and R the radius of the curvature) required to obtain a particular value of Nusselt number is less for visco-elastic fluid than that required for a Newtonian fluid.

1. INTRODUCTION

The evaluation of the effect of curvature of a curved pipe on the heat transfer rate is of fundamental importance for the design of heating or cooling coils. Frass and Ozisik (1965) have detailed out that the heat transfer in curved pipes or helical coils play fundamental role in various heat exchangers. Metzner (1965) and Skelland (1967) have reviewed in detail the heat transfer problems for Non-Newtonian fluids and their applications. Mori and Nakayama (1965; 1967*a*,*b*) have studied the temperature distribution for large L for Newtonian fluids. Recently, Ozisik and Topakoglu (1968) have obtained theoretically the temperature distribution for Newtonian fluids in curved pipe for small L which are in good agreement with experimental observations made by Seban and McLaughlin (1963).

In the present paper, we have considered the heat transfer effects in a curved pipe for visco-elastic fluid. The case of Newtonian fluid is deduced

from the analysis of visco-elastic fluid. Following Dean (1927, 1928), we assume that the radius of curvature of the pipe is large as compared to the radius of curvature cross-section of pipe. The velocity field is known in advance from the analysis of Thomas and Walters (1963). The temperature profiles are evaluated up to first order of L, where L is defined to be as $2W_0^2a^3/\nu^2R$. For the evaluation of Nusselt number, approximation is carried up to second order of L.

Following Ozisik and Topakoglu (1968), we introduce the following assumptions :

(i) Velocity and temperature fields are fully developed.

(ii) Flow is laminar.

(iii) Viscous dissipation is negligible.

(iv) Wall temperature around the periphery of any cross-section and its rate of change in the axial direction are uniform.

2. Formulation of the Problem

Following Dean (1928), Fig. 1 shows the system of coordinates used in the analysis. C is the centre of the section of pipe by a plane that makes an angle θ with the fixed axial plane. CO is perpendicular drawn from C upon OX and is of length R, the radius of curvature of the pipe. The position of any point P in the section is specified by the orthogonal coordinates

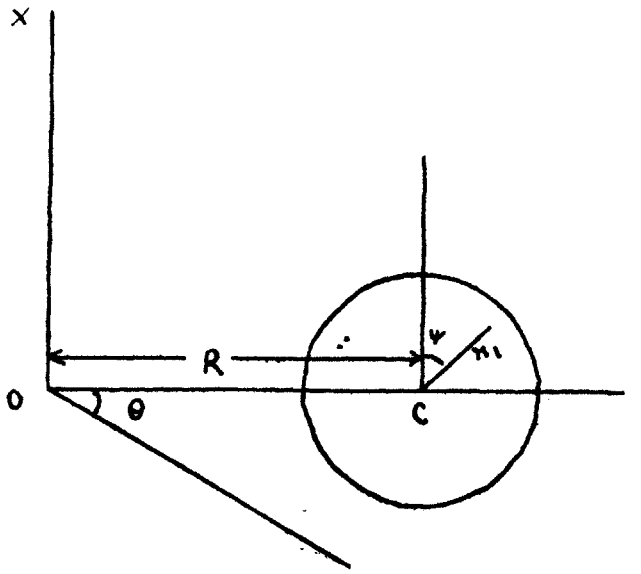

Fig. 1. The coordinate system

(r_1, ψ, θ) where $CP = r_1$ and ψ is the angle that CP makes with a line through C and parallel to OX. The same coordinate system has been used by Thomas and Walters (1963) for determining the velocity field of a visco-elastic fluid. Following the assumption made in the introduction, the temperature at any point in the cross-section has the following form :

$$T = MR\theta + N(r_1, \psi) \quad (N \text{ is const. at the wall}) \qquad \ldots \quad (1)$$

where M is a constant which determines the temperature gradient in the longitudinal direction of the pipe. We will assume N to be zero over the wall so that $N(r_1, \psi)$ denotes the excess temperature.

Introducing the assumptions of Dean (1928), the energy equation can be written in the form :

$$U\frac{\partial N}{\partial r_1} + \frac{V}{r_1}\frac{\partial N}{\partial \psi} + WM = \chi\left[\frac{\partial^2 N}{\partial r_1^2} + \frac{1}{r_1}\frac{\partial N}{\partial r_1} + \frac{1}{r_1^2}\frac{\partial^2 N}{\partial \psi^2}\right] \qquad \ldots \quad (2)$$

where U, V and W are velocity components in the directions of r_1, ψ and θ respectively and χ is the thermal diffusivity.

Introducing the non-dimensional variables :

$$U = \frac{\nu}{a}u, \quad V = \frac{\nu}{a}v, \quad W = W_0 w, \quad r_1 = ar, \quad M = \frac{E}{k}m, \quad N = \frac{aE}{k}n \qquad \ldots \quad (3)$$

where u, v, w have the same meaning as given by Thomas and Walters (1963) and m, n and non-dimensional temperature gradient and excess temperature respectively. E and k are the reference flux and the conductivity respectively.

Substituting (3) in (2), we have

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \psi^2}\right]n = P_r\left[u\frac{\partial n}{\partial r} + \frac{v}{r}\frac{\partial n}{\partial \psi}\right] + P_r R_e w m \qquad \ldots \quad (4)$$

where $P_r = \nu/\chi$ (Prandtl number) and $R_e = aW_0/\nu$ (Reynolds number).

Following Thomas and Walters (1963), we assume

$$\begin{aligned} n &= n_0 + Ln_1 + L^2 n_2 + \ldots \\ u &= Lu_1 + L^2 u_2 + \ldots \\ v &= Lv_1 + L^2 v_2 + \ldots \\ w &= w_0 + Lw_1 + L^2 w_2 + \ldots \end{aligned} \qquad \ldots \quad (5)$$

Substituting (5) in (4) and comparing coefficients of L^0, we have

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \psi^2}\right]n_0 = P_r R_e m w_0 \qquad \ldots \quad \ldots \quad (6)$$

where $w_0 = 1 - r^2$ as given by Thomas and Walters (1963).

The solution of differential equation (6) satisfying boundary conditions that $n_0 = 0$ at $r = 1$ and n_0 is finite at $r = 0$, is given by

$$n_0 = -\frac{P_r R_e m}{16}[1-r^2][3-r^2]. \quad \ldots \quad \ldots \quad (7)$$

From (5) and (4), comparing coefficients of L, we have

$$\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \psi^2}\right] n_1 = P_r R_e m w_1 + P_r u_1 \frac{\partial n_0}{\partial r} \quad \ldots \quad (8)$$

where n_0 is given by (7).

Following Thomas and Walters (1963), we write

$$u_1 = -\frac{1}{r}\frac{\partial \phi_1}{\partial \psi}, \qquad v_1 = \frac{\partial \phi_1}{\partial r} \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

where ϕ_1 is given as

$$\phi_1 = \left[\left(\frac{1}{144}+\frac{1}{24}S\right) r - \left(\frac{1}{64}+\frac{1}{12}S\right) r^3 + \left(\frac{1}{96}+\frac{1}{24}S\right) r^5 - \frac{r^7}{576}\right]$$

$$\cos \psi = g_1 \cos \psi \quad \ldots \quad (10)$$

and

$$w_1 = \left[\left(\frac{19}{40}+11S+48S^2\right) r - (1+24S+96S^2) r^3 + \left(\frac{3}{4}+16S+48S^2\right) r^5 - \left(\frac{1}{4}+3S\right) r^7 + \frac{r^9}{40}\right] \frac{\sin \psi}{576} = h_1 \sin \psi \quad \ldots \quad (11)$$

where S is the visco-elastic parameter.

With the help of (9), we write (8) as

$$\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \psi^2}\right] n_1 = P_r R_e m w_1 - P_r \cdot \frac{1}{r} \cdot \frac{\partial n_0}{\partial r} \cdot \frac{\partial \phi_1}{\partial \psi} \quad \ldots \quad (12)$$

(10), (11) and (12) suggest the form of n_1 as

$$n_1 = l_1 \sin \psi \quad \ldots \quad \ldots \quad (13)$$

where l_1 is a function of r. The differential equation determining l_1, is given by

$$\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1}{r^2}\right] l_1 = \frac{P_r R_e m}{576}\left[\left(\frac{19}{40}+11S+48S^2\right) r - (1+24S+96S^2) r^3 + \left(\frac{3}{4}+16S+48S^2\right) r^5 - \left(\frac{1}{4}+3S\right) r^7 + \frac{r^9}{40}\right] + \frac{P_r^2 R_e m}{4r}\left[\left(\frac{1}{144}+\frac{1}{24}S\right) r - \left(\frac{1}{64}+\frac{1}{12}S\right) r^3 + \left(\frac{1}{96}+\frac{1}{24}S\right) r^5 - \frac{r^7}{576}\right][2r-r^3]. \quad \ldots \quad (14)$$

The solution of eqn. (14) satisfying the boundary condition that l_1 is zero at $r = 1$ and l_1 is finite at $r = 0$, is given by

$$l_1 = \frac{P_r R_e m}{576}\left[A_1 r + G_1 r^3 + G_2 r^5 + G_3 r^7 + G_4 r^9 + \frac{r^{11}}{4800}\right]$$

$$+ \frac{P_r R_e m}{576}\left[B_1 r + G_5 r^3 + G_6 r^5 + G_7 r^7 + G_8 r^9 + \frac{r^{11}}{120}\right] \qquad (15)$$

where A_1, B_1, G_1, ..., G_8 are functions of S.

From (5) and (4), comparing coefficients of L^2, we have

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \psi^2}\right] n_2 = P_r R_e m w_2 + P_r\left[\frac{v_1}{r}\frac{\partial n_1}{\partial r} + u_2\frac{\partial n_0}{\partial r} + u_1\frac{\partial n_1}{\partial r}\right]. \qquad \ldots (16)$$

Following Thomas and Walters (1963), we write

$$u_2 = -\frac{1}{r}\frac{\partial \phi_2}{\partial \psi}, \qquad v_2 = \frac{\partial \phi_2}{\partial r} \qquad \ldots \quad \ldots (17)$$

where ϕ_2 is given as

$$\phi_2 = J_1 \sin 2\psi \quad \text{and} \quad w_2 = \bar{w}_2 + \bar{w}_{22} \cos 2\psi \qquad \ldots \quad \ldots (18)$$

From (16) and (17), we have

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \psi^2}\right] n_2 = P_r R_e m w_2 + \frac{P_r}{r}\left[\frac{\partial \phi_1}{\partial r}\cdot\frac{\partial n_1}{\partial \psi} - \frac{\partial \phi_2}{\partial \psi}\cdot\frac{\partial n_0}{\partial r}\right.$$

$$\left. - \frac{\partial \phi_1}{\partial \psi}\cdot\frac{\partial n}{\partial r_1}\right] \qquad \ldots \quad \ldots (19)$$

(18) and (19) suggest the form of n_2 as

$$n_2 = l_{20} + l_{22} \cos 2\psi. \qquad \ldots \quad \ldots (20)$$

Since, for calculating mean bulk temperature, contribution of l_{22} would not be there due to presence of $\cos 2\psi$ as multiplier, we confine our attention to l_{20} only. Thus, with the help of (18) and (20), (19) takes the following form :

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right] l_{20} = P_r R_e \bar{w} w_2 + P_r\left[\frac{1}{r}\cdot\frac{d}{dr}(\tfrac{1}{2} g_1 l_1)\right]. \qquad \ldots \quad \ldots (21)$$

The solution of eqn. (21) satisfying the boundary conditions that $l_{20} = 0$ at $r = 1$ and l_{20} is finite at $r = 0$, is given as

$$l_{20} = \frac{P_r R_e m}{2\times(576)^2}\left[H_1+H_2r^2+H_3r^4+H_4r^6+H_5r^8+H_6r^{10}+H_7r^{12}\right.$$

$$\left.+H_8r^{14}+H_9r^{16}-\frac{0{\cdot}0016}{324}r^{18}\right]+\frac{P_r^2R_em}{2\times(576)^2}\times\left[H_{10}+H_{11}r^2+H_{12}r^4\right.$$

$$\left.+H_{13}r^6+H_{14}r^8+H_{15}r^{10}+H_{16}r^{12}+H_{17}r^{14}+H_{18}r^{16}-\frac{1}{18\times4800}r^{18}\right]$$

$$+\frac{P_r^3R_em}{2\times(576)^2}\left[H_{19}+H_{20}r^2+H_{21}r^4+H_{22}r^6+H_{23}r^8+H_{24}r^{10}\right.$$

$$\left.+H_{25}r^{12}+H_{26}r^{14}+H_{27}r^{16}-\frac{1}{72\times120}r^{18}\right]. \quad \ldots (22)$$

where $H_1, \ldots, H_{27}$ are functions of S whose forms are too complicated to be given here.

3. Nusselt Number

The mean bulk temperature N_b is defined as

$$N_b = \frac{\rho}{Q}\int_0^{2\pi}\int_0^{a} NW dr_1\, r_1\, d\psi \quad \ldots \quad \ldots (23)$$

where ρ is the density and Q is the mass flow rate.

When the pipe is straight, we have the mean bulk temperature N_{0b} as

$$N_{0b} = -\frac{2\pi\rho a^3 E W_0}{Qk}\cdot\frac{11}{24\times16}. \quad \ldots \quad \ldots (24)$$

Substituting for N, W and r_1, from (3) in (23), we have

$$N_b = \frac{\rho}{Q}\int_0^{2\pi}\int_0^{1}\frac{aE}{k}\cdot nW_0 w a^2 r dr d\psi. \quad \ldots \quad \ldots (25)$$

Substituting for n and w from (5) in (25), we have

$$N_b = \frac{\rho}{Q}\int_0^{2\pi}\int_0^{1}\frac{aE}{k}\, W_0[n_0+Ln_1+L^2n_2+\ldots]\,[w_0+Lw_1+L^2w_2+\ldots]\, a^2 r dr d\psi. \quad \ldots (26)$$

With the help of (11), (13), (18), (20), (24), and neglecting terms of order higher than L^2, (26) takes the followigng form :

$$N_b = \frac{Q_0}{Q}N_{0b} + \frac{2\pi\rho a^3 E W_0}{Qk}\int_0^{1} rL^2[n_0\bar{w}_2+w_0l_{20}+\tfrac{1}{2}l_1h_1]dr. \quad \ldots (27)$$

From (24) and (27), we have

$$\frac{N_b}{N_{0b}} = \frac{Q_0}{Q}\left[1+\frac{L^2\int_0^1 r[n_0\bar{w}_2+w_0l_{20}+\frac{1}{2}l_1h_1]dr}{\int_0^1 rn_0w_0dr}\right] \quad \ldots \quad \ldots (28)$$

where $\frac{Q_0}{Q}$, the flux ratio is given by Thomas and Walters (1963)

as $$\frac{Q}{Q_0} = \left[1-\left(\frac{L}{576}\right)^2[0{\cdot}3058-0{\cdot}064265S]\right]. \quad \ldots (29)$$

Nusselt number for a straight pipe may be defined as

$$N_{0u} = \frac{2a}{k}\cdot\frac{Q_0}{N_{0b}}. \quad \ldots \quad \ldots (30)$$

If Nu is Nusselt number of a curved pipe, we have

$$\frac{Nu}{N_{0u}} = \frac{11}{48}Nu = \frac{Q}{B_b}\cdot\frac{N_{0b}}{Q_0} = \left(\frac{Q}{Q_0}\right)^2\Bigg/\left[1+\frac{L^2\int_0^1 r[n_0\bar{w}_2+w_0l_{20}+\frac{1}{2}l_1h_1]dr}{\int_0^1 rn_0w_0dr}\right] \quad \ldots (31)$$

where

$$\int_0^1 r[n_0\bar{w}_2+w_0l_{20}+\tfrac{1}{2}l_1h_1]dr = P_rR_em[K_1+P_rK_2+P_rK_3] \quad \ldots (32)$$

where K_1, K_2, K_3 are functions of S whose forms are too complicated to be given here.

4. Numerical Example

Isotherms in the Plane of Cross-section of Pipe

From (5), to a first order of approximation, we have

$$n = n_0+Ln_1 = n_0+Ll_1\sin\psi \quad \ldots \quad \ldots (33)$$

where n_0 and l_1 are functions of r only. For $S = 0$ the isotherms for Newtonian fluid are shown in Fig. 2. The isotherms are plotted for Prandtl number 100 and $L = 100$. For the purpose of illustration, the isotherms are plotted for values $10^{-2}n$ = constant. We notice that the isotherms are similar to the ones given by Hawes (1932) based on his experimental observations as confirmed by Ozisik and Topakoglu (1968). Figs. 3, 3*a* and 3*b* show the isotherms for the case of visco-elastic fluid for $S = 1$, 0·1, 0·05 respectively.

The general pattern of the isotherms for the visco-elastic fluid is similar to that of Newtonian fluid. Isotherms are in the form of closed curves and the

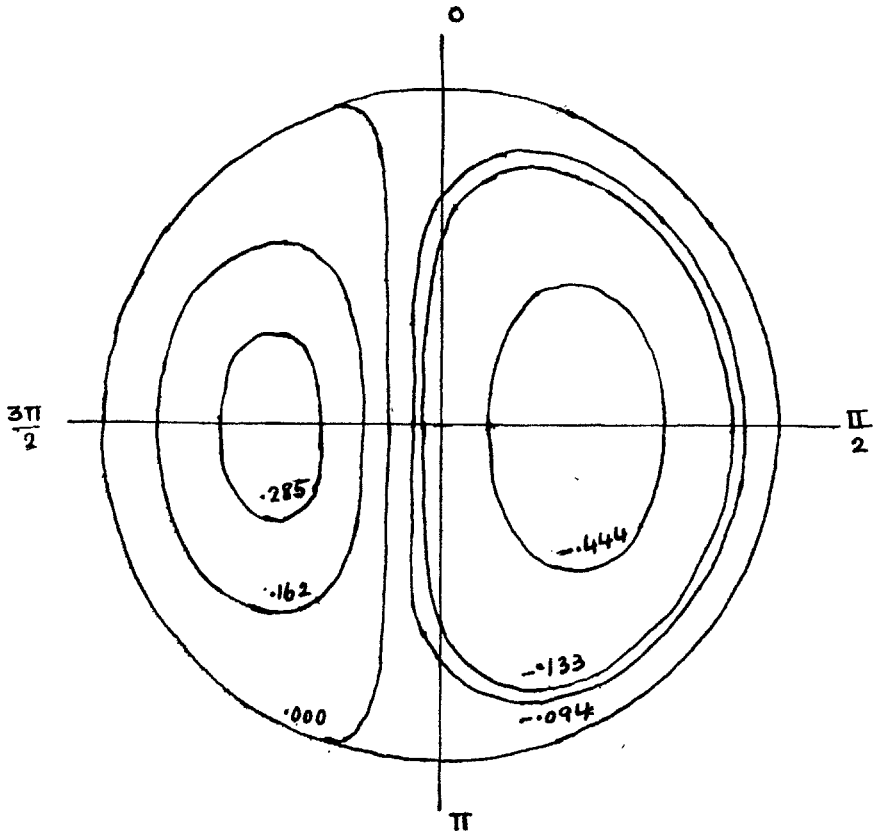

Fig. 2. Isotherms for Prandtl Number 100, $L = 100$ $S = 0$

cross-section is divided in two regions by curve $n = 0$. The domain towards the outer side of curvature represents the cooled region, while the inner side represents the heated region. The following table shows the maximum drop and maximum rise in temperature for different values of S. Barnes

S	Max. Rise in Temp.	Max. Drop in Temp.
0·0	0·372	0·656
0·05	0·537	0·819
0·1	0·703	0·983
0·2	1·038	1·317
0·6	2·407	2·686
1·0	3·826	4·105

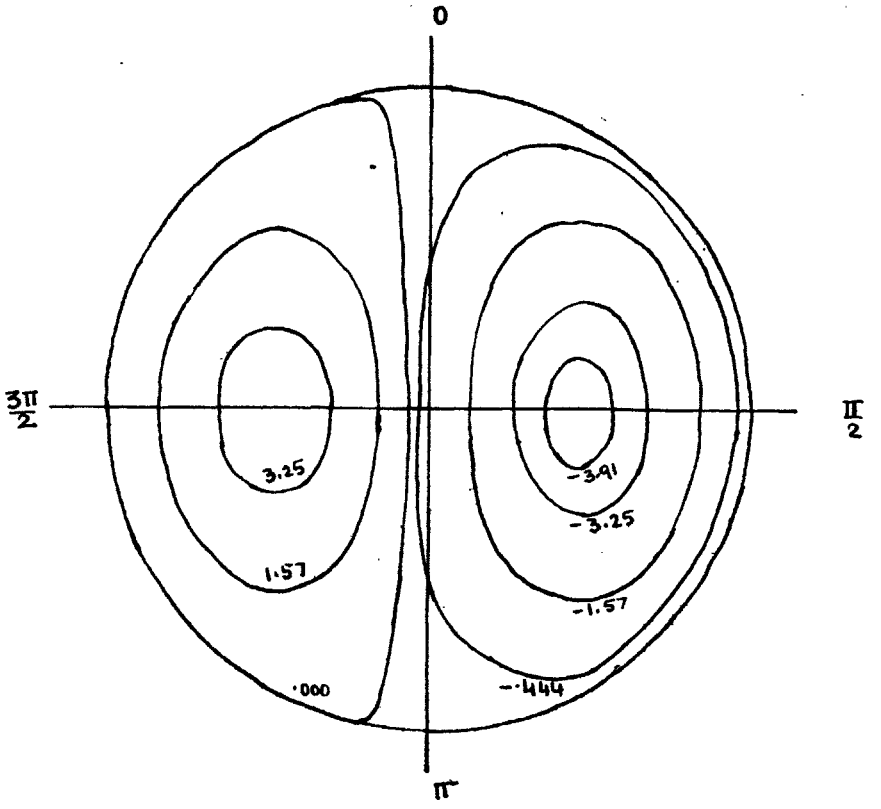

Fig. 3. Isotherms for Prandtl Number 100, $L = 100$ $S = 1$.

and Walters (1969) have argued that increasing S may be interpreted loosely as increasing the elasticity in the fluid. The convective effects are, therefore, expected to be more as S is increased producing higher and lower temperatures.

5. Discussion of Nusselt Number

Equation (31) shows that the Nusselt number for a curved pipe depends on two parameters : namely Parndtl number, L, visco-elastic parameter S. For Prandtl number 500, 100 we have plotted N_u/N_{0u} agaisnt L for various values of S (Figs. 4 and 5). The curves for $S = 0$ show that the results agree well with the theoretical analysis of Ozisik and Topakoglu (1968) whose results are in good agreement with the experimental observations made by Seban and McLaughlin (1963). The curves both for Newtonian and visco-elastic fluid show that the Nusselt number for a curved pipe is valid for a limited small values of L for a particular Prandtl number. Also we find that

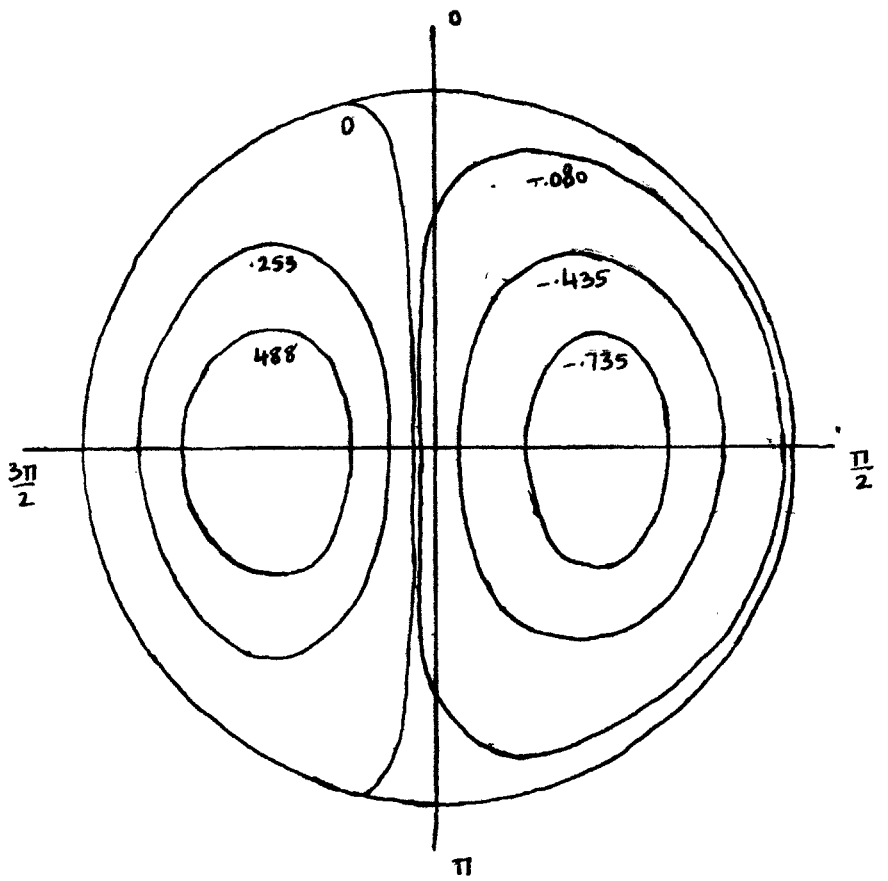

Fig. 3a. Isotherms for $P_r=100$, $L=100$, $S=\cdot 1$.

as the Prandtl number is decreased, the range of L for which Nusselt number for curved pipe differs from that of a straight pipe is increased. The curves also show that Nusselt number for a curved pipe does not deviate from that of a straight pipe until L reaches a certain value.

From the curves shown in Figs. 4 and 5, it is observed that due to the presence of elasticity the value of L required to attain a particular value of N_u/N_{0u} for visco-elastic fluid is less than that required for the case of Newtonian fluid.

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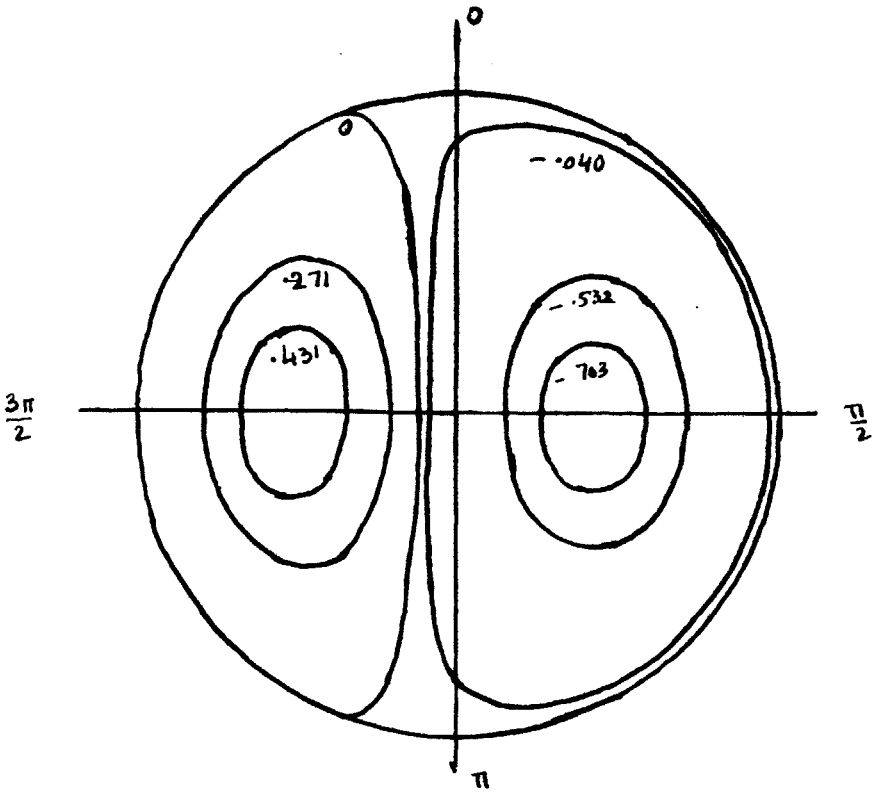

Fig. 3b. Isotherms for $P_r = 100$, $L = 100$, $S = \cdot 05$.

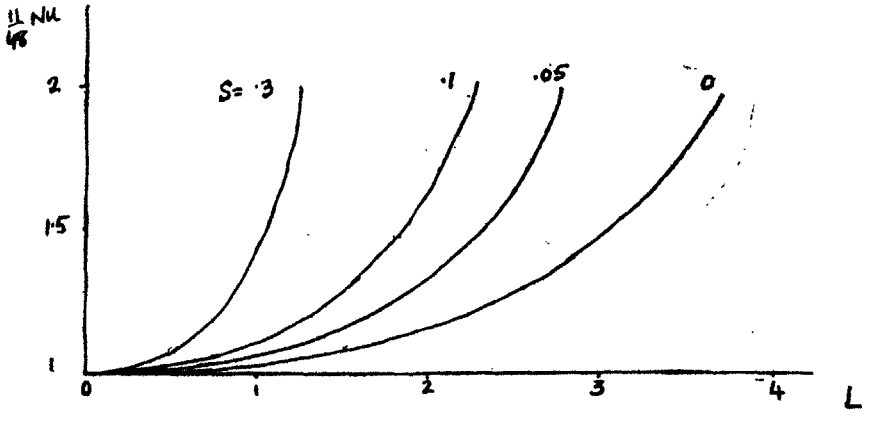

Fig. 4. Nusselt number versus parameter L for different S [$P_r = 500$].

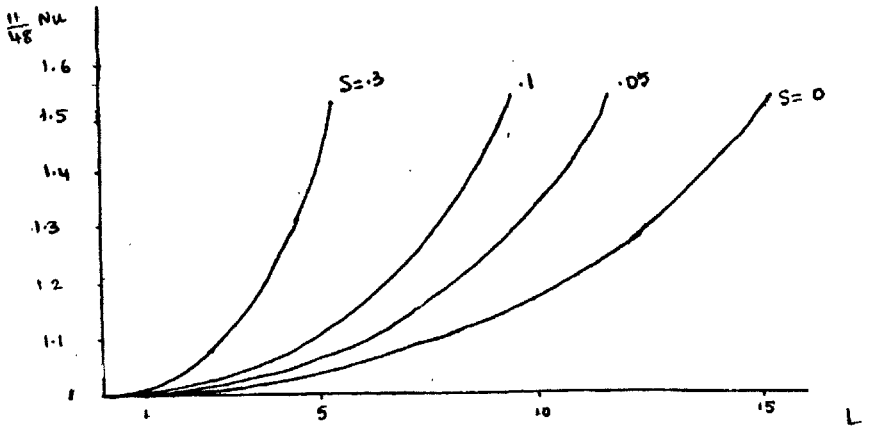

Fig. 5. Nusselt number versus parameter L for different values of S [$Pr = 100$].

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