DYNAMIC THERMOELASTIC RESPONSE OF A SLAB UNDER PRESCRIBED HEAT INPUT

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The method of differential operators has been used for finding the thermoelastic stresses, displacements and the temperature in an infinite slab of constant finite thickness when one face of the slab is rigidly constrained and the other is assumed to be stress-free. At the stress-free face there is a prescribed heat input and the other face is maintained at constant temperature. The effect of inertia has been considered in the equations of motion but the mechanical coupling term in the heat conduction equation has been omitted. The general solution is obtained by applying Mittag-Leffler theorem to the solutions in terms of differential operators. The variation of temperature, heat input, stresses and displacements have been shown grraphically.

1. INTRODUCTION

In this paper we study the thermoelastic stresses in an infinite flat slab which arise due to a prescribed heat input at one face of the slab. We take the slab to be elastically and thermally isotropic and make the small-strain, linear elasticity assumptions, The effect of inertia in the equation of motion has been taken into account. The linear heat conduction equation and the equations of elasticity are solved for the temperature, displacements and the stresses in the slab by 'Symbolic Method' in terms of differential operators (Lurye 1955 and Lekhnitskii 1963). The general solution to the thermoelastic problem has been obtained by the application of Mittag-Leffler theorem (*see* Whittaker and Watson 1962). The operators of the type $f(q)/g(q)$, $q = \partial/\partial t$, are decomposed into rational fractions and the solution for the thermoelastic boundary value problem is obtained in terms of simple integrals in an elegant form. Numerical results are obtained by expanding the solution in the operator form as infinite series in terms of derivatives of the prescribed function of time and the approximate results are obtained by truncating the series after a few terms.

Several papers have appeared in the literature on problem similar to this. The method of differential operators has been used by Wadhawan and Singh (1970) for finding the steady-state thermo-elastic stresses in a slab under prescribed surface temperature. Martin and Payton (1964) and Martin (1966) used the theory of complex variables and Fourier transform in the solution of a similar problem. Other references may be found in the paper of Sneddon and Lockett (1960).

2. STATEMENT OF THE PROBLEM

Using the rectangular cartesian system of co-ordinates x, y, z, let the infinite slab, bounded by two parallel planes $x = 0$ and $x = L$, be infinite in y and z directions. The face $x = 0$ is held rigid and is maintained at zero temperature, while the face $x = L$, taken to be stress free, is exposed to a continuous heat input varying with time t only.

The displacement components u, v, w in the x, y, z directions respectively are

$$u = u(x, t), \quad v = w = 0. \qquad \ldots \quad (2.1)$$

The equations of elasticity here are ((Sokolnikoff 1956) :

$$\frac{\partial \sigma_{xx}}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2}, \qquad \ldots \quad (2.2)$$

$$\sigma_{xx} = \frac{E}{(1+\nu)(1-2\nu)} \left[(1-\nu) \frac{\partial u}{\partial x} - \alpha(1+\nu)T \right], \qquad \ldots \quad (2.3)$$

$$\sigma_{yy} = \sigma_{zz} = \frac{E}{(1+\nu)(1-2\nu)} \left[\nu \frac{\partial u}{\partial x} - \alpha(1+\nu)T \right], \qquad \ldots \quad (2.4)$$

$$\tau_{xy} = \tau_{yz} = \tau_{zx} = 0. \qquad \ldots \quad (2.5)$$

where $(\sigma_{xx}, \sigma_{yy}, \sigma_{zz})$ and $(\tau_{xy}, \tau_{yz}, \tau_{zx})$ are components of the normal and shear stress respectively. Following the usual convention, ρ, E, ν, α and T are respectively the density, Young's modulus, Poisson's ratio, the coefficient of linear thermal expansion and the temperature distribution in the slab.

The appropriate boundary and initial conditions are

$$\left.\begin{array}{l} u(0, t) = 0, \quad t \geqslant 0 \\ \sigma_{xx}(L, t) = 0, \quad t \geqslant 0 \\ u = \sigma_{xx} = \sigma_{yy} = \sigma_{zz} = 0, \quad t < 0 \end{array}\right\} \qquad \ldots \quad (2.6)$$

The heat conduction equation here is

$$\frac{\partial^2 T}{\partial x^2} = \frac{\rho c}{k}\frac{\partial T}{\partial t}, \quad t > 0 \qquad \ldots \quad (2.7)$$

where k and c are respectively the thermal conductivity and specific heat of the solid.

The thermal boundary conditions are

$$\left.\begin{aligned} &T(0, t) = 0, \quad t \geqslant 0, \\ &k\frac{\partial T}{\partial x} = \phi(t), \quad \text{at } x = L, \end{aligned}\right\} \qquad \ldots \quad (2.8)$$

where $\phi(t)$ is the heat input which is a continuous or piecewise continuous function of t.

Introducing the dimensionless quantities

$$T^* = \frac{T}{T_R}, \qquad x^* = \frac{x}{L}, \qquad t^* = \frac{kt}{\rho c L^2}, \qquad u^* = \frac{u}{L} \qquad \ldots \quad (2.9)$$

where T_R is any reference temperature, the form of eqns. (2.2) to (2.5) remains unchanged and eqns. (2.7) and (2.8) may be re-written as

$$\left.\begin{aligned} &\frac{\partial^2 T^*}{\partial x^{*2}} = \frac{\partial T^*}{\partial t}, \quad 0 < x^* < 1, \quad t^* > 0 \\ &T^*(0, t^*) = 0, \quad t^* \geqslant 0 \\ &\frac{\partial T^*}{\partial x^*} = \phi^*(t^*) \quad \text{at } x^* = 1. \end{aligned}\right\} \qquad \ldots \quad (2.10)$$

In the subsequent discussion, we shall drop the asterisks.

3. Solution of the Problem in the Operator Form

The solution of the heat conduction equation, on using the thermal boundary conditions is

$$T = \frac{\sinh x\sqrt{q}}{\sqrt{q}\cosh\sqrt{q}}\,\phi(t), \quad q \equiv \frac{\partial}{\partial t}. \qquad \ldots \quad (3.1)$$

Writing eqns. (2.2) and (2.3) in the operator form, we obtain after simplification,

$$(D_x^2 - mq^2)u = \frac{l\cosh x\sqrt{q}}{\cosh\sqrt{q}}\,\phi(t) \qquad \ldots \quad (3.2)$$

where T_R is taken as the unit of temperature for simplicity and

$$D_x \equiv \frac{\partial}{\partial x}, \quad l = \frac{\alpha(1+\nu)}{(1-\nu)}, \quad m = \frac{k^2(1+\nu)(1-2\nu)}{E(1-\nu)\rho c^2 L^2}. \qquad \ldots \quad (3.3)$$

Solving eqn. (3.2) for u by applying the boundary conditions (2.6), u is obtained in the following form :

$$u = \frac{l}{\Delta}[\sinh \sqrt{m}xq(\sinh \sqrt{mq} - \sqrt{mq} \sinh \sqrt{q}) + \cosh \sqrt{mq}$$
$$\times(\cosh x\sqrt{q} - \cosh \sqrt{m}xq)]\phi(t), \qquad \ldots \quad (3.4)$$

where

$$\Delta \equiv (1-mq)q \cosh \sqrt{mq} \cosh \sqrt{q}. \qquad \ldots \quad (3.5)$$

Substituting the values of u and T in eqns. (2.3) and (2.4), the stresses are obtained in the following form :

$$\frac{(1-2\nu)}{E\alpha}\sigma_{xx} = \frac{1}{\Delta}[\sqrt{mq} \cosh \sqrt{m}xq\,(\sinh \sqrt{mq} - \sqrt{mq} \sinh \sqrt{q})$$
$$-\sqrt{mq}\,(\sinh \sqrt{m}xq - \sqrt{mq} \sinh x\sqrt{q}) \cosh \sqrt{mq}]\,\phi(t) \ \ldots \quad (3.6)$$

$$\frac{(1-\nu)(1-2\nu)}{\nu E\alpha}\sigma_{yy} = \frac{1}{\Delta}\Big[\sqrt{mq} \cosh \sqrt{m}xq\,(\sinh \sqrt{mq} - \sqrt{mq} \sinh \sqrt{q})$$
$$+\left\{\frac{1}{\nu}(mq - \nu mq + 2\nu - 1)\sqrt{q} \sinh \sqrt{q}x - \sqrt{mq} \sinh \sqrt{m}xq\right\}$$
$$\times \cosh \sqrt{mq}\Big]\,\phi(t). \qquad \ldots \quad (3.7)$$

Equations (3.1), (3.4), (3.6) and (3.7) constitute the general solution to the given thermoelastic boundary value problem in the operator form. The function $\phi(t)$ has to be prescribed to get the numerical results.

4. SOLUTION BY MITTAG-LEFFLER THEOREM

The expressions (3.1), (3.4), (3.6) and (3.7) for the temperature T, displacement u and the normal stresses σ_{xx}, σ_{yy} in terms of the differential operator q are of the form $[F(q)]\phi(t)$. The function $F(q)$, as a function of the complex variable q, satisfies the conditions of Mittag-Leffler theorem and can, therefore, be expressed in rational fractions of the form

$$F(q) = F(0) + \sum_{m=1}^{n} b_m \left\{\frac{1}{q-a_m} + \frac{1}{a_m}\right\} \qquad \ldots \quad (4.1)$$

where $a_1, a_2, \ldots, a_n$ are simple poles of $F(q)$ and $b_1, b_2, \ldots, b_n$ are the residues at these simple poles.

In the case of temperature T, the function $F(q)$ has simple poles at $q = -\frac{1}{4}(2n+1)^2\pi^2$, $n = 0, 1, 2, 3, \ldots$, and is analytic at the origin. The value of $F(0)$ and residues at $-\frac{1}{4}(2n+1)^2\pi^2$ in this case are

$$\left.\begin{aligned} F(0) = x, \quad \operatorname{Res} F(q) &= 2(-1)^n \sin (2n+1)\frac{\pi}{2}x \\ \text{at} \quad q &= -\frac{(2n+1)^2}{4}\pi^2 \end{aligned}\right\} \qquad \ldots \quad (4.2)$$

Using Mittag-Leffler theorem, the solution for temperature T takes the form :

$$T = x\phi(t) + 2\sum_{n=0}^{\infty}(-1)^n \sin (2n+1)\frac{\pi}{2}x \left[e^{-(2n+1)\frac{\pi^2}{4}t} \times \right.$$

$$\left. \int_0^t e^{(2n+1)^2\frac{\pi^2}{4}\tau}\phi(\tau)d\tau - \frac{4}{\pi^2(2n+1)^2}\phi(t) \right]. \qquad \ldots \quad (4.3)$$

The corresponding function $F(q)$ for the displacement u and the stresses σ_{xx}, σ_{yy} is analytic at the origin and has simple poles :

$$\frac{1}{m}, \quad \pm\frac{(2n+1)\pi}{2\sqrt{m}}i, \quad -(2n+1)^2\frac{\pi^2}{4}, \quad n = 0, 1, 2, 3, \ldots \qquad \ldots \quad (4.4)$$

For the displacement u, we have

$$F(0) = \operatorname*{Lt}_{q \to 0} F(q) = \tfrac{1}{2}x^2. \qquad \ldots \quad (4.5)$$

The residues at the simple poles using the expression (3.4) for the displacement u are obtained as :

$$\left.\begin{aligned} &\operatorname{Res} F(q) = 0, && \text{at} \quad q = \frac{1}{m}, \\ &\operatorname{Res} F(q) = H(x, n) && \text{at} \quad q = (2n+1)\frac{\pi}{2\sqrt{m}}i, \end{aligned}\right\} \qquad \ldots \quad (4.6)$$

$$\operatorname{Res} F(q) = \overline{H}(x, n) \quad \text{at} \quad q = -\frac{(2n+1)\pi i}{2\sqrt{m}} \Big\}, \qquad \ldots \quad (4.7)$$

where

$$\left. H(x, n) = (-1)^n \sin x\gamma_n \frac{[\sin \gamma_n - \sqrt{m}\, S_n(1+i) \sin (1-i)\, S_n]}{\gamma_n(1 - \sqrt{m}\,\gamma_n i \cos (1-i)\, S_n} \right\} \qquad \ldots \quad (4.8)$$

$$S_n^2 = \frac{(2n+1)\pi}{4\sqrt{m}}, \qquad \gamma_n = (2n+1)\frac{\pi}{2}$$

and $\overline{H}(x, n)$ is the complex conjugate of $H(x, n)$.

$$\left.\operatorname{Res} F(q) = G(x, n) \qquad \text{at} \qquad q = -(2n+1)^2 \frac{\pi^2}{4} \right\} \qquad \ldots \quad (4.9)$$

where

$$\left.\begin{aligned} G(x, n) &= 2(-1)^{n-1} \\ &\times \frac{[\cosh \beta_n \cosh \gamma_n x - (-1)^n \sqrt{m}\, \gamma_n \sinh \beta_n \gamma_n \sqrt{m}\, x - \cosh \beta_n (x-1)]}{(1+\sqrt{m}\, \beta_n)\, \gamma_n \cosh \beta_n} \\ \text{and} \qquad \beta_n &= \sqrt{m}(2n+1)^2 \frac{\pi^2}{4}. \end{aligned}\right\} \quad \ldots \quad (4.10)$$

The displacement u is obtained in the following form :

$$\begin{aligned} \frac{u}{l} = \frac{x^2}{2} \phi(t) + \sum_{n=0}^{\infty} \Big[& H(x, n) e^{i\gamma_n' t} \int_0^t e^{-i\gamma_n' \tau} \phi(\tau) d\tau \\ & + \overline{H}(x, n) e^{-i\gamma_n' t} \int_0^t e^{i\gamma_n' \tau} \phi(\tau) d\tau + \frac{i}{\gamma_n'} (\overline{H}(x, n) - H(x, n)) \phi(t) \Big] \\ & + \sum_{n=0}^{\infty} \left[G(x, n) \left\{ e^{-\gamma_n^2 t} \int_0^t e^{\gamma_n^2 \tau} \phi(\tau) d\tau - \frac{1}{\gamma_n^2} \phi(t) \right\} \right] \quad \ldots \quad (4.11) \end{aligned}$$

where $\quad \gamma_n' = \dfrac{\gamma_n}{\sqrt{m}}.$

The stresses σ_{xx} and σ_{yy} are obtained as :

$$\begin{aligned} \frac{(1+\nu)(1-2\nu)}{(1-\nu)El} \sigma_{xx} = \sum_{n=0}^{\infty} \Big[\Big\{ & H_x(x, n) e^{i\gamma_n' t} \int_0^t e^{-i\gamma_n' \tau} \phi(\tau) d\tau \\ & + \overline{H}_x(x, n) e^{-i\gamma_n' t} \int_0^t e^{i\gamma_n' \tau} \phi(\tau) d\tau + \frac{i}{\gamma_n'} (\overline{H}_x(x, n) - H_x(x, n)) \phi(t) \Big\} \Big] \\ & + \sum_{n=0}^{\infty} G_x(x, n) \left\{ e^{-\gamma_n^2 t} \int_0^t e^{\gamma_n^2 \tau} \phi(\tau) d\tau - \frac{1}{\gamma_n^2} \phi(t) \right\} \Big] \\ & - 2 \sum_{n=0}^{\infty} (-1)^n \sin \gamma_n x \left[e^{-\gamma_n^2 t} \int_0^t e^{\gamma_n^2 \tau} \phi(\tau) d\tau - \frac{1}{\gamma_n^2} \phi(t) \right] \quad \ldots \quad (4.12) \end{aligned}$$

$$\begin{aligned} \frac{(1+\nu)(1-2\nu)}{E} \sigma_{yy} = l\nu \Big[x\phi(t) + \sum_{n=0}^{\infty} \Big\{ & H_x(x, n) e^{i\gamma_n' t} \int_0^t e^{-i\gamma_n' \tau} \phi(\tau) d\tau \\ & + \overline{H}_x(x, n) e^{-i\gamma_n' t} \int_0^t e^{i\gamma_n' \tau} \phi(\tau) d\tau + \frac{i}{\gamma_n'} (\overline{H}_x(x, n) - H_x(x, n)) \phi(t) \Big\} \end{aligned}$$

$$+\sum_{n=0}^{\infty} G_x(x, n) \left\{ e^{-\gamma_n{}^2 t} \int_0^t e^{\gamma_n{}^2 \tau} \phi(\tau) d\tau - \frac{1}{\gamma_n^2} \phi(t) \right\} \Bigg]$$

$$-\alpha(1+\nu) \Bigg[x\phi(t) + 2 \sum_{n=0}^{\infty} (-1)^n \sin \gamma_n x$$

$$\times \left\{ e^{-\gamma_n{}^2 t} \int_0^t e^{\gamma_n{}^2 \tau} \phi(\tau) d\tau - \frac{\phi(t)}{\gamma^2{}_n} \right\} \Bigg] \qquad \ldots \quad (4.13)$$

where $H_x(x, n)$ and $\overline{H}_x(x, n)$ denote derivatives with respect to x. It is clear that the expressions (4.11), to (4.13) for the displacement u and the stresses respectively reduce to real quantities.

The integrals occurring in expressions for displacement and the stresses given by (4.11) to (4.13) can be solved if the function $\phi(t)$ is prescribed. This completes the general solution to the given problem.

5. NUMERICAL EXAMPLE

The expressions in (3.1), (3.4), (3.6) and (3.7) are expanded as infinite series in the form

$$\left[\sum_{n=0}^{\infty} \beta_n q^n \right] \phi(t) \qquad \ldots \quad (5.1)$$

where β_n are functions of x only and $q^n \equiv \dfrac{\partial^n}{\partial t^n}$.

The sequence $\{\beta_n\}$ is seen to be rapidly converging as is observed from actual calculations. For numerical calculations we take

$$\phi(t) = \frac{e^{-bt}}{(1+at)}, \quad 0 < b < 1, \quad a > 0,$$

and we set $b = 0{\cdot}5$ and $a = 0{\cdot}1$. We take steel as the material of the slab with $\nu = 0{\cdot}17$, $\rho = 7{\cdot}86$ gm/cm^3, $E = 17{\cdot}2 \times 10^{11}$ dynes/cm^2, $\alpha = 11{\cdot}7 \times 10^{-6}$.

It is found that the terms in (5.1) form a rapidly converging series and the first six terms of (5.1) give reasonably good approximation.

The expression for the temperature T and the displacement u in the form of equation (5.1) is as follows :

$$T = x + \left(\frac{x^3}{3!} - \frac{x}{2!} \right) q + \left(\frac{x^5}{5!} - \frac{x^3}{3!2!} + \frac{5}{4!} x \right) q^2$$

$$+ \left(\frac{x^7}{7!} - \frac{x^5}{5!\,2!} + \frac{5x^3}{3!\,4!} - \frac{61x}{6!} \right) q^3 +$$

$$+\left(\frac{x^9}{9!}-\frac{x^7}{7!\,2!}+\frac{5x^5}{5!\,4!}-\frac{61x^3}{3!\,6!}+\frac{1385}{8!}\,x\right)q^4$$

$$+\left(\frac{x^{11}}{11!}-\frac{x^9}{9!\,2!}+\frac{5x^7}{7!\,4!}-\frac{61x^5}{5!\,6!}+\frac{1385}{3!\,8!}\,x^3-\frac{50521}{10!}\,x\right)q^5+\ldots\Big]\,\phi(t) \qquad (5.2)$$

$$\frac{u}{l}=\frac{x^2}{2!}\,\phi(t)+\left[\frac{x^4}{4!}-\frac{x^2}{(2!)^2}\right]q\phi(t)+\Big[\frac{x^6}{6!}+\frac{x^2m}{(2!)^2}-\frac{xm}{3!}+\left(m+\frac{1}{2!}\right)\left(\frac{x^4}{4!}-\frac{x^2m}{2!}\right)$$

$$+\left(m^2-m-\frac{5}{4!}\right)\frac{x^2}{2!}\Big]\,q^2\phi(t)+\Big[\frac{x^8}{8!}-\frac{x^4m^2}{4!}+\frac{m}{2!}\left(\frac{x^4}{4!}-\frac{x^2m}{2!}\right)+xm\left(\frac{m}{3!}-\frac{1}{5!}\right)$$

$$+\left(m-\frac{1}{2!}\right)\left(\frac{x^6}{6!}+\frac{x^2m}{(2!)^2}-\frac{xm}{3!}\right)+\left(m^2-m-\frac{5}{4!}\right)\left(\frac{x^4}{4!}-\frac{x^2m}{2!}\right)$$

$$+\left(m^3-m^2+\frac{11}{4!}\,m-\frac{61}{6!}\right)\frac{x^2}{2!}\Big]\,q^3\phi(t)+\Big[\frac{x^{10}}{10!}+\frac{x^6m}{2!\,6!}+\frac{x^2m^2}{2!\,4!}-xm\left(\frac{x^2m}{(3!)^2}+\frac{1}{7!}\right)$$

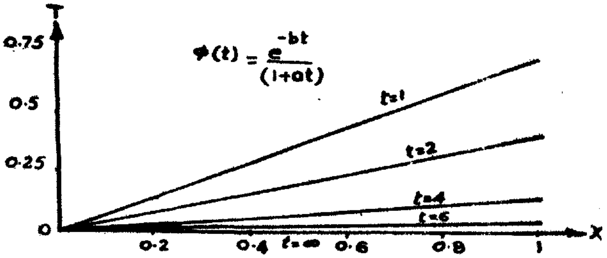
$$+\left(m-\frac{1}{2!}\right)\left\{\frac{x^8}{8!}-\frac{x^4m^2}{4!}+\frac{m}{2!}\left(\frac{x^4}{4!}-\frac{x^2m}{2!}\right)+xm\left(\frac{m}{3!}-\frac{1}{5!}\right)\right\}$$

$$+\left(m^2-m-\frac{5}{4!}\right)\left(\frac{x^6}{6!}+\frac{x^2m}{(2!)^2}-\frac{xm}{3!}\right)+\left(m^3-m^2+\frac{11m}{4!}-\frac{61}{6!}\right)\left(\frac{x^4}{4!}-\frac{x^2m}{2!}\right)$$

$$+\left(m^4-m^3+\frac{2}{3}\,m^2-\frac{136m}{6!}+\frac{1385}{8!}\right)\frac{x^2}{2!}\Big]\,q^4\phi(t)+\ldots \qquad \ldots \quad (5.3)$$

Similar expressions for σ_{xx}, σ_{yy} and $\partial T/\partial x$ are not reproduced here to conserve space.

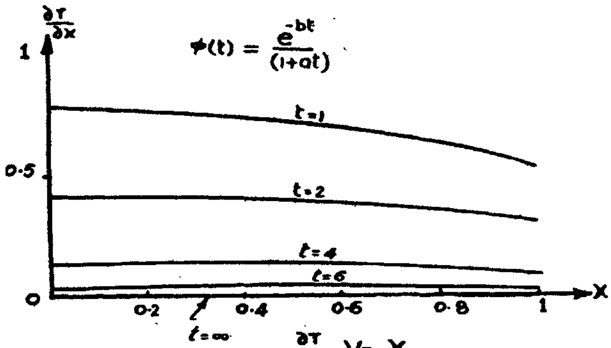
A series solution of the given thermoelastic problem is obtained which is very suitable for numerical calculations. This series converges if the successive derivatives of the prescribed heat input function form a converging sequence but a broader set of conditions can be obtained for the convergence of this infinite series.



T Vs X

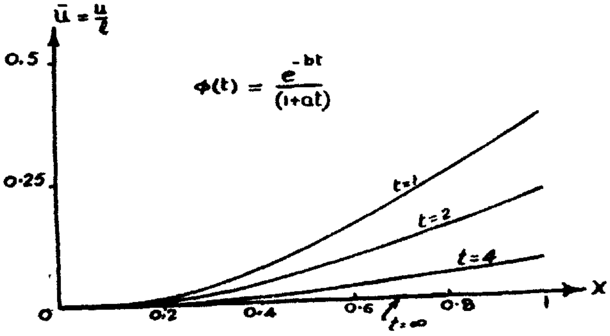
FIG 1

The variations of the temperature, heat input, displacement and the stresses are shown graphically in Figs. 1 to 5 when $\nu = 0{\cdot}17$ snd $t = 1, 2, 4, 6, \infty$.



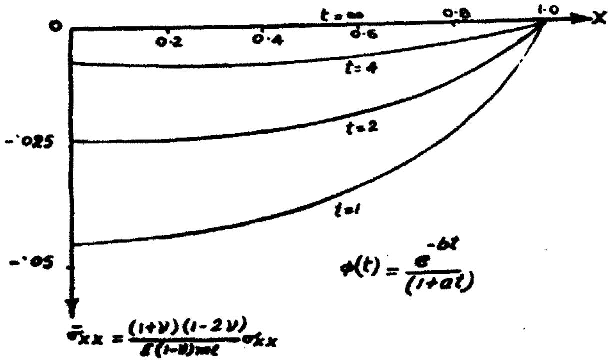
$\frac{\partial T}{\partial x}$ Vs X

FIG. 2



ū Vs x

FIG. 3



$\bar{\sigma}_{xx}$ Vs x

FIG. 4

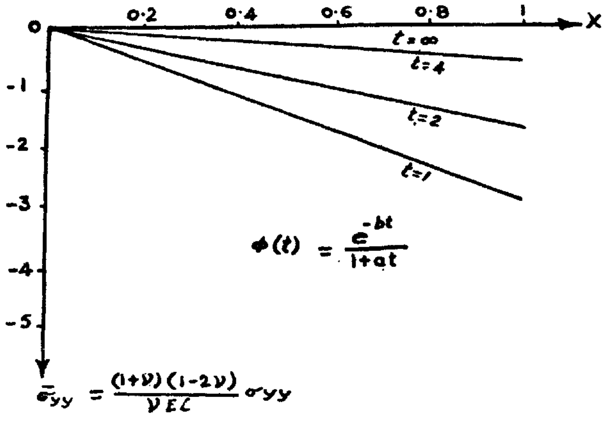


FIG. 5

The temperature in the slab increases as x increases from 0 to 1 but steadily decreases as t increases and when $t \to \infty$, the temperature is zero. The displacement u increases monotonically with x but reduces to zero as t tends to infinity.

Figures 4 and 5 plot the normal stresses σ_{xx} and σ_{yy} respectively. The stress σ_{xx} is only due to the effect of inertia and consequently its magnitude is extremely small. At the face $x = 0$ of the slab, the stress σ_{yy} is due to the inertia effects only and is neglected in the figure. It is noted that these stresses are compressions at every point in the slab.

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