ON THE NÖRLUND SUMMABILITY OF THE CONJUGATE DERIVED SERIES OF THE FOURIER SERIES

by L. M. TRIPATHI *and* K. PRASAD, *Department of Mathematics, Banaras Hindu University, Varanasi* 5

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In the present paper we have established a theorem which is a generalization of the theorem for harmonic summability of conjugate derived series of the Fourier Series (Tripathi 1963).

1. If $\sum_{n=0}^{\infty} a_n$ is a series, we shall use the notation

$$S_n = \sum_{r=0}^{n} a_r. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.1)$$

Let $\{p_n\}$ be a sequence with $p_0 > 0$ and $p_n \geqslant 0$ for $n > 0$.

For α real, we define

$$P_n^{(\alpha)} = \sum_{r=0}^{n} p_r^{\alpha} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.2)$$

where

$$P_n = \sum_{r=0}^{n} p_r. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.3)$$

Definition (Cass 1969)—Nörlund summability (N, p_n^{α}). For $\alpha > -1$ and $\sum_{r=0}^{\infty} a_r$ a series, let

$$t_n^{(\alpha)} = \frac{1}{P_n^{(\alpha)}} \sum_{r=0}^{n} p_{n-r}^{\alpha} S_r. \quad \ldots \quad \ldots \quad \ldots \quad (1.4)$$

If $t_n^{(\alpha)} \to S$ as $n \to \infty$ we write

$$\sum_{r=0}^{\infty} a_r = S(N, p_n^{\alpha}) \quad \text{or} \quad S_n \to S(N, p_n^{\alpha}).$$

2. Let $f(t)$ be a periodic function with period 2π and integrable in the sense of Lebesgue over the interval $(-\pi, \pi)$.

Let its Fourier series be

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt). \quad \ldots \quad \ldots \quad \ldots \quad (2.1)$$

The series

$$\sum_{n=1}^{\infty} n(b_n \cos nt - a_n \sin nt) \quad \ldots \quad \ldots \quad \ldots \quad (2.2)$$

which is obtained by differentiating (2.1) term by term is called the first derived series or the derived Fourier series of $f(t)$.

The series conjugate to (2.2) is

$$\sum_{n=1}^{\infty} n(a_n \cos nt + b_n \sin nt). \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2\ 3)$$

We write

$$h(t) = f(x+t)+f(x-t)-2f(x)$$

$$\chi(t) = \int_0^t |dh(u)|$$

$$\tau = \left[\frac{1}{t}\right]$$

where $[t]$ denotes the integral part of t.

We define

$$H_n(x) = -\frac{1}{4\pi}\int_{\frac{1}{n}}^{\pi} h(t)\operatorname{cosec}^2 \tfrac{1}{2}t\, dt$$

$$H(x) = -\frac{1}{4\pi}\int_0^{\pi} h(t)\operatorname{cosec}^2 \tfrac{1}{2}t\, dt$$

so that as $n \to \infty$, $H_n(x) \to H(x)$.

3. Young (1943) and Riesz (1923) were the first mathematicians to use derived series and conjugate derived series systematically. Later on, Mohanty (1950), Mohanty and Nanda (1955), Chow (1955) and a number of other researchers studied the (c, α) summability of the above mentioned series. Tripathi (1963) established the following theorem for the harmonic summability of the conjugate derived series of a Fourier series.

Theorem—The conjugate derived series of the Fourier series of the function $f(x)$ is summable by harmonic means to the sum

$$-\frac{1}{4\pi}\int_0^{\pi} h(t)\operatorname{cosec}^2 \tfrac{1}{2}t\, dt$$

at every point x at which this integral exists and

$$\chi(t) = O(t/\log 1/t), \quad \text{as } t \to 0.$$

The object of the present paper is to study the conjugate derived series of Fourier series by (N, p_n^{α}) summability method which is more comprehensive than that of Harmonic. As a matter of fact (N, p_n^{α}) summability method includes as special cases both the Harmonic and Cesaro methods of summation. Thus our theorem will extend the above theorem for the more general summability method of Nörlund. More precisely we prove the following theorem.

Theorem—A Nörlund method of Summation (N, p_n^{α}) defined by a real, non-negative, non-increasing sequence of coefficients $\{p_n^{\alpha}\}$ such that $P_n^{(\alpha)} \to \infty$ as $n \to \infty$, sums the conjugate derived series of the Fourier series of the function $f(x)$ to the sum

$$-\frac{1}{4\pi}\int_0^{\pi} h(t)\operatorname{cosec}^2 \tfrac{1}{2}t \, dt \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.1)$$

at every point x at which this integral exists and

$$\chi(t) = 0(p_{\tau}^{(\alpha)}/P_{\tau}^{(\alpha)}).$$

4. The following lemmas are required for the proof of the theorem

Lemma 1—For $\nu < n$

$$\left|\int_0^{\frac{1}{n}} \cot \tfrac{1}{2}t(1-\cos \nu t)dh(t)\right| = 0(1).$$

Proof : We have

$$\left|\int_0^{\frac{1}{n}} \cot \tfrac{1}{2}t(1-\cos \nu t)dh(t)\right| \leqslant \int_0^{\frac{1}{n}}\left|\frac{\cos \frac{1}{2}t}{\sin \frac{1}{2}t} \; 2\sin^2 \frac{\nu t}{2}\right| |dh(t)|$$

$$\leqslant 2\nu \int_0^{\frac{1}{n}} |dh(t)|$$

$$= 2\nu\chi\left(\frac{1}{n}\right)$$

$$= 0(\nu p_u^{\alpha}/P_n^{(\alpha)})$$

$$= 0(1).$$

Lemma 2—For $\nu < n$

$$-\frac{1}{2\pi}\int_{\frac{1}{n}}^{\pi} \cot \tfrac{1}{2}t(1-\cos \nu t)dh(t) = \frac{1}{2\pi}\int_{\frac{1}{n}}^{\pi} \cot \tfrac{1}{2}t \cos \nu t \, dh(t) + H_n(x) + 0(1).$$

Proof : We have

$$-\frac{1}{2\pi}\int_{\frac{1}{n}}^{\pi} \cot \tfrac{1}{2}t(1-\cos \nu t)dh(t)$$

$$= -\frac{1}{2\pi}\int_{\frac{1}{n}}^{\pi} \cot \tfrac{1}{2}t\, dh(t)+\frac{1}{2\pi}\int_{\frac{1}{n}}^{\pi} \cot \tfrac{1}{2}t \cos \nu t\, dh(t)$$

$$= -\frac{1}{2\pi}\Big[\cot \tfrac{1}{2}t\, h(t)\Big]_{\frac{1}{n}}^{\pi} - \frac{1}{2\pi}\int_{\frac{1}{n}}^{\pi} \tfrac{1}{2} \operatorname{cosec}^2 \tfrac{1}{2}t\, h(t)dt+\frac{1}{2\pi}\int_{\frac{1}{n}}^{\pi}\cot \tfrac{1}{2}t \cos \nu t\, dh(t).$$

$$= \frac{1}{2\pi}\,\frac{\cot\left(\frac{1}{2n}\right)\left(\frac{1}{n}\right) h\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} + H_n(x)+\frac{1}{2\pi}\int_{\frac{1}{n}}^{\pi} \cot \tfrac{1}{2}\, t \cos \nu t\, dh(t)$$

$$= 0(1)+H_n(x)+\frac{1}{2\pi}\int_{\frac{1}{n}}^{\pi} \cot \tfrac{1}{2}\, t \cos \nu t\, dh(t)$$

for $h(t)/t \to 0$ with t, as $f'(x)$ exists.

Lemma 3—If $\{p_n^{\alpha}\}$ is non-negative and non-increasing then for $0 \leqslant a < b \leqslant \infty$, $0 \leqslant t \leqslant \pi$ and any n

$$\left|\sum_{k=a}^{b} p_k^{\alpha} e^{i(n-k)t}\right| \leqslant A P_{\tau}^{(\alpha)}$$

where A is an absolute constant.

The proof of the Lemma follows on the lines of McFadden (1942).

Lemma 4—If $\frac{1}{n} \leqslant t \leqslant \delta < \pi$, then

$$\bar{N}_n(t) \equiv \frac{1}{2\pi P_n^{(\alpha)}} \sum_{k=0}^{n} p_k^{\alpha} \frac{\cos (n-k+\frac{1}{2})t}{\sin \frac{1}{2}\, t}$$

$$= O\left(\frac{P_{\tau}^{(\alpha)}}{t\, P_n^{(\alpha)}}\right).$$

Proof : We have

$$|\bar{N}_n(t)| = \frac{1}{2\pi P_n^{(\alpha)}}\left|\sum_{k=0}^{n} p_k^{\alpha} \frac{\cos(n-k+\frac{1}{2})t}{\sin \frac{1}{2}\, t}\right|$$

$$\leqslant \frac{1}{2\pi P_n^{(\alpha)}|\sin \frac{1}{2}t|}\left| \operatorname{Re} \sum_{k=0}^{n} p_k^{\alpha} \exp(i(n-k+\tfrac{1}{2})t\right|$$

$$= \frac{1}{2\pi P_n^{(\alpha)} |\sin \frac{1}{2}t|} \left| \text{Re} \left(e^{i\frac{1}{2}t} \sum_{k=0}^{n} p_k^{\alpha} \exp\left(i(n-k)t\right) \right) \right|$$

$$\leqslant \frac{1}{2\pi P_n^{(\alpha)}} \left| \sum_{k=0}^{n} p_k^{\alpha} \exp\left(i(n-k)t\right) \right|$$

$$= O(P_\tau^{(\alpha)}/t\, P_n^{(\alpha)}), \quad \text{by Lemma 3.}$$

Proof of the Theorem—Let $\bar{\sigma}_n(x)$ denote the sum of the first n terms of the series (2.3) at $t = x$. Then we have

$$\bar{\sigma}_n(x) = -\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \frac{\partial}{\partial u} \left(\sum_{1}^{n} \sin \nu(u-x) du \right)$$

$$= -\frac{1}{\pi} \int_0^{\pi} \frac{d}{dt} \left[\frac{\cos \frac{1}{2}t - \cos(n+\frac{1}{2})t}{\sin \frac{1}{2}t} \right] [f(x+t)+f(x-t)]dt$$

$$= -\frac{1}{\pi} \int_0^{\pi} \frac{\cos \frac{1}{2}t - \cos(n+\frac{1}{2})t}{\sin \frac{1}{2}t} \, dh(t).$$

Hence

$$\bar{\sigma}_\nu(x) = -\frac{1}{2\pi} \int_0^{\pi} [\cot \tfrac{1}{2}t(1-\cos \nu t) + \sin \nu t] dh(t), \; \nu < n.$$

$$= -\frac{1}{2\pi} \left\{ \int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^{\pi} \right\} \cot \tfrac{1}{2}t(1-\cos \nu t) dh(t) - \frac{1}{2\pi} \int_0^{\pi} \sin \nu t \, dh(t).$$

$$= -\frac{1}{2\pi} \int_{\frac{1}{n}}^{\pi} \cot \tfrac{1}{2}t \cos \nu t \, dh(t) - \frac{1}{2\pi} \int_0^{\pi} \sin \nu t \, dh(t) + H_n(x) + 0(1)$$

by Lemma 1 and 2.

$$= \frac{1}{2\pi} \int_{\frac{1}{n}}^{\pi} \cot \tfrac{1}{2}t \cos \nu t \, dh(t) - \frac{1}{2\pi} \int_{\frac{1}{n}}^{\pi} \sin \nu t \, dh(t) - \frac{1}{2\pi} \int_0^{\frac{1}{n}} \sin \nu t \, dh(t) + H_n(x) + 0(1)$$

$$= \frac{1}{2\pi} \int_{\frac{1}{n}}^{\pi} \frac{\cos(\nu+\frac{1}{2})t}{\sin \frac{1}{2}t} \, dh(t) + H_n(x) = 0(1).$$

Therefore,

$$\bar{\sigma}_{n-k}(x) - H_n(x) = \frac{1}{2\pi} \int_{\frac{1}{n}}^{\pi} \frac{\cos(n-k+\frac{1}{2})t}{\sin \frac{1}{2}t} \, dh(t) + 0(1).$$

Applying (1.4) and denoting $\frac{1}{P_n^{(\alpha)}} \sum_{k=0}^{n} p_k^{\alpha} [\bar{\sigma}_{n-k}(x) - H_n(x)]$ by $\bar{Z}_n(t)$ we get

$$\bar{Z}_n(t) = \frac{1}{P_n^{(\alpha)}} \sum_{k=0}^{n} p_k^{\alpha} \left\{ \frac{1}{2\pi} \int_{\frac{1}{n}}^{\pi} \frac{\cos(n-k+\frac{1}{2})t}{\sin \frac{1}{2}t} dh(t) + 0(1) \right\}$$

$$= \int_{\frac{1}{n}}^{\pi} dh(t) \frac{1}{2\pi P_n^{(\alpha)}} \left\{ \sum_{k=0}^{n} p_k^{\alpha} \frac{\cos(n-k+\frac{1}{2})t}{\sin \frac{1}{2}t} \right\} + 0(1)$$

$$= \int_{\frac{1}{n}}^{\pi} dh(t) \bar{N}_n(t) + 0(1).$$

In order to prove the theorem we have to show that under assumptions

$$\bar{Z}_n(t) = 0(1) \text{ as } n \to \infty \text{ when (3.1) exists.} \quad \dots \quad \dots \quad (4.1)$$

We write

$$\bar{Z}_n(t) = \left\{ \int_{\frac{1}{n}}^{\delta} + \int_{\delta}^{\pi} \right\} dh(t) \bar{N}_n(t) + 0(1)$$

$$= I_1 + I_2 + 0(1), \text{ say} \quad \dots \quad \dots \quad \dots \quad (4.2)$$

when $0 < \delta < \pi$.

New by Lemma 4, we have

$$I_1 = O\left[\int_{\frac{1}{n}}^{\delta} |dh(t)|\, |\bar{N}_n(t)| \right]$$

$$= O\left[\frac{1}{P_n^{(\alpha)}} \int_{\frac{1}{n}}^{\delta} |dh(t)| \frac{P_\tau^{(\alpha)}}{t} \right]$$

$$= O\left[\frac{1}{P_n^{(\alpha)}} \left\{ (\chi t) \frac{P_\tau^{(\alpha)}}{t} \right\}_{\frac{1}{n}}^{\delta} - \frac{1}{P_n^{(\alpha)}} \int_{\frac{1}{n}}^{\delta} \chi(t) \left\{ -\frac{P_\tau^{(\alpha)}}{t^2} dt + \frac{1}{t} dP_\tau^{(\alpha)} \right\} \right]$$

$$= O\left[\frac{1}{P_n^{(\alpha)}} \left\{ \chi(t) \frac{P_\tau^{(\alpha)}}{t} \right\}_{\frac{1}{n}}^{\delta} + \frac{1}{P_n^{(\alpha)}} \int_{\frac{1}{n}}^{\delta} \chi(t) \frac{P_\tau^{(\alpha)}}{t^2} dt - \frac{1}{P_n^{(\alpha)}} \int_{\frac{1}{n}}^{\delta} \chi(t) \frac{1}{t} dP_\tau^{(\alpha)} \right]$$

$$= O[J_1 + J_2 + J_3], \text{ say} \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (4.3)$$

Now $$J_1 = \frac{1}{P_n^{(\alpha)}} \left[\frac{\chi(d) P^{(\alpha)}_{(\frac{1}{\delta})}}{\delta} \right] - \frac{1}{P_n^{(\alpha)}} \left[\chi\left(\frac{1}{n}\right) \cdot \frac{P_n^{(\alpha)}}{1/n} \right]$$

$$= 0\left(\frac{1}{P_n^{(\alpha)}} \right) - 0\left(\frac{n\, p_n^{\alpha}}{P_n^{(\alpha)}} \right)$$

$$= 0(1) + 0(1)$$

$$= 0(1) \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (4.4)$$

$$J_2 = 0\left[\frac{1}{P_n^{(\alpha)}}\int_{\frac{1}{n}}^{\delta}\frac{p_\tau^\alpha}{P_\tau^{(\alpha)}}\frac{P_\tau^{(\alpha)}}{t^2}\,dt\right]$$

$$= 0\left[\frac{1}{P_n^{(\alpha)}}\int_{\frac{1}{n}}^{\delta}\frac{p_\tau^\alpha}{t^2}\,dt\right]$$

$$= 0\left[\frac{1}{P_n^{(\alpha)}}\int_{\frac{1}{\delta}}^{n}p_{[s]}\,ds\right]$$

$$= 0\left[\frac{1}{P_n^{(\alpha)}}\sum_{k=0}^{n}p_k^\alpha\right]$$

$$= 0(1) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.5)$$

$$J_3 = 0\left[\frac{1}{P_n^{(\alpha)}}\int_{\frac{1}{n}}^{\delta}\frac{p_\tau^\alpha}{P_\tau^{(\alpha)}}\frac{1}{t}\,dP_\tau^{(\alpha)}\right]$$

$$= 0\left[\frac{1}{P_n^{(\alpha)}}\int_{n}^{\frac{1}{\delta}}\frac{s\,p_{[s]}^\alpha}{P_{[s]}^{(\alpha)}}\,dP_{[s]}^{(\alpha)}\right]$$

$$= 0\left[\frac{1}{P_n^{(\alpha)}}\int_{\frac{1}{\delta}}^{n}dP_{[s]}^{(\alpha)}\right]$$

$$= 0\left[\frac{1}{P_n^{(\alpha)}}\sum_{k=0}^{n}p_k^\alpha\right]$$

$$= 0(1). \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.6)$$

Lastly, by virtue of Riemann-Lebesgue theorem we have

$$I_2 = O\left\{\int_{\delta}^{\pi}dh(t)\bar{N}_n(t)\right\}$$

$$= 0(1) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.7)$$

as $n \rightarrow \infty$.

Combining (4.2) to (4.7) we see that (4.1) holds.

This completes the proof of the theorem.

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