EFFECT OF A WEAK SWIRL ON A STEADY SUPERSONIC GAS FLOW IN A CIRCULAR TUBE

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The paper deals with supersonic flow of an inviscid gas under isentropic change of state through a circular tube in the presence of a weak swirl. At the entry section of the tube the axial velocity and temperature are assumed to be uniform and the azimuthal velocity corresponds to a wheel flow. The purpose of the present investigation is to see how this initial simple flow evolves along the length of the tube. The investigation predicts a decrease in mass flow rate, positive excess of the wall-temperature over the axial temperature and very small deviation from the initial azimuthal velocity distribution. The radial velocity remains small but shows considerable variation with the radial distance at all cross-sections. The results are qualitatively in agreement with the experimental observations of Laven and Fejer (1965). However, a quantitative comparison between the results of the present investigation and those of Laven and Fejer is not possible as the perturbation method adopted here restricts the initial circumferential velocity to be small. However, with the increasing values of this velocity, the results show a trend in the right direction.

1. INTRODUCTION

Since the discovery of vortex tube by Ranque (1933) a large number of investigations have been made on the vortical flows and extensive bibliographies (Westley 1954, Gartshore 1962, Dobratz 1964) exist which give references to these investigations. We therefore, refer only to the excellent experimental investigation of Laven and Fejer (1965) to which the present investigation owes its inspiration. For a Mach-2 nozzle, Laven and Fejer observed the following important facts : (i) the azimuthal flow near the axis of the cylindrical tube conforms to a solid-body rotation while near the wall it has the character of a free-vortex, (ii) the temperature and pressure increase from the axis to the wall, (iii) the presence of swirl produces decrease in the mass flow rate which led them to suggest that a swirl may be used to control the mass flow in a supersonic pipe flow and (iv) in the presence of a

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comparatively large swirl, a luminescent core surrounding the axis is produced which remained more or less unaffected by an increase or decrease in the entry Mach number. Evidently, the fourth observation cannot be considered in the present paper as the phenomenon is attributed to the production of electric charge on the water droplets present in the gas and the consequent discharge.

The present paper considers a steady laminar flow of an inviscid gas in a circular tube under isentropic change of state when the axial velocity is supersonic and the swirl is weak. We assume that at the entry section axial velocity, density and temperature are uniform and the rotation conforms to a solid-body rotation in view of the observation (i) in the investigation of Laven and Fejer though this observation strictly relates to the core surrounding the axis. The purpose of this investigation is to see how this simple initial flow in the entry section evolves with axial distance. We have used the perturbation technique assuming

$$\epsilon = \frac{\text{circumferential velocity}}{\text{axial velocity}}$$

in the entry section as a small parameter. The various order equations are linear and suitable for solution by Laplace transform method with respect to axial coordinate. We have considered the approximation up to ϵ^4 but even the fourth order perturbation solution involves triple summation, which could be effected only with the help of a high speed computer. The method adopted here has restricted the admissible values of ϵ to small magnitudes not exceeding 0·08. We have arrived at this value from the consideration that the fourth order perturbation is at least one order smaller than the earlier. This restriction, though appears to be very severe, in fact allows us to study the circumferential velocities up to $M\epsilon$, where M is the entry Mach number for axial velocity. The present investigation does indicate an increase in temperature and a decrease in axial velocity from the axis to the swall from certain axial distance onwards. Moreover, these variations in temperature and axial velocity profiles become more and more pronounced as ϵ increases but remain much smaller than the observed ones on account of the restriction on ϵ mentioned above. The radial velocity always remains small but shows considerable variation in its character. The azimuthal velocity shows very small deviation from the initial solid-body rotation at all axial distances showing thereby that most significant interaction takes place between axial velocity and temperature fields through the agency of radial and azimuthal velocity fields.

Besides the reference cited above, there are some recent investigations which are in some ways related to the theme of the present paper. For example, Talbot (1954) and Binnie (1957) have discussed the laminar swirling flows of a viscous 'incompressible' fluid in a cylindrical tube mainly to evaluate the azimuthal and axial velocity fields. Pangelley (1957), Rott (1958), Gartshore (1962), Reynolds (1961*a*, *b*) and Subulkin (1962) have discussed a variety of vortical flow problems when the motion is 'turbulent' and the fluid is 'incompressible'. Keyes (1960), Rogsdale (1960), and Kerrenbrock and Meghreblian (1961) have studied the motion of a compressible fluid in a vortex tube with a strong swirl and a weak axial velocity in an effort to contribute towards development of fission rockets. Gambell *et al.* (1961) and Kreith and Sonjn (1965) have experimentally studied the turbulent swirling flows in tubes with tape inserts with a view to improve heat exchanger performances.

2. Governing Equations

Let us consider a perfect gas entering a circular tube of radius a at the end $z = 0$ with uniform axial velocity w_0, density ρ_0, and temperature T_0 undergoing a solid-body rotation with angular velocity Ω, so that the azimuthal velocity of gas at $z = 0$ is given by $v_0 = \gamma\Omega$. Let us take a, w_0, ρ_0, T_0, $\frac{1}{2}\rho_0 w_0^2$ as characteristic length velocity, density, temperature and pressure and denote the non-dimensional radial, azimuthal and axial velocity components by u, v, w, and non-dimensional density pressure and temperature by ρ, p and T. The governing equations for a steady inviscid Laminar axisymmetric flow in non-dimensional form are :

Momentum Equations

$$uu_r + wu_z - \frac{1}{r}v^2 = -\frac{1}{\rho}p_r = -\frac{A\gamma}{\gamma-1}\frac{\partial}{\partial r}(\rho^{\gamma-1}) \qquad \ldots \quad (2.1)$$

$$uv_r + wv_z + \frac{1}{r}uv = 0 \qquad \ldots \quad (2.2)$$

$$uw_r + ww_z = -\frac{1}{\rho}p_z = -\frac{A\gamma}{\gamma-1}\frac{\partial}{\partial z}(\rho^{\gamma-1}) \qquad \ldots \quad (2.3)$$

Equation of Continuity

$$\rho\left(u_r + \frac{1}{r}u + w_z\right) + u\rho_r + w\rho_z = 0 \qquad \ldots \quad (2.4)$$

'Adiabatic' Relation

$$p = A\rho^{\gamma} \qquad \ldots \quad (2.5)$$

where

$$A\gamma = \frac{2}{M^2}, \quad M = \frac{w_0}{a_0}, \quad \gamma = C_p/C_v \qquad \ldots (2.6)$$

M being the entry Mach number for the axial velocity and $a_0 = (A\gamma\rho_0^{\gamma-1})^{\frac{1}{2}}$ the sound speed.

Perfect Gas Equation

In view of (2.5) the perfect gas equation reduces to

$$T = \rho^{\gamma-1}. \qquad \ldots (2.7)$$

We now make the assumption of weak swirl in the form

$$\frac{a\Omega}{w_0} = \epsilon < 1$$

and expand the flow variables in powers of ϵ in the following form as suggested by the governing equations :

$$\begin{aligned} u &= \epsilon^2 u_2 + \epsilon^4 u_4 + \ldots \\ v &= \epsilon v_1 + \epsilon^3 v_3 + \ldots \\ w &= 1 + \epsilon^2 w_2 + \epsilon^4 w_4 + \ldots \\ \rho &= 1 + \epsilon^2 \rho_2 + \epsilon^4 \rho_4 + \ldots \\ T &= 1 + \epsilon^2 T_2 + \epsilon^4 T_4 + \ldots \end{aligned} \qquad \ldots (2.8)$$

We can easily check that the following equations determine the various order approximations to flow variables :

First order : $$v_{1z} = 0. \qquad \ldots (2.9)$$

Second order : $$u_{2z} + A\gamma\rho_{2r} = \frac{1}{r} v_1^2 \qquad \ldots (2.10)$$

$$(w_2 + A\gamma\rho_2)_z = 0 \qquad \ldots (2.11)$$

$$\frac{1}{r}(ru_2)_r + (w_2 + \rho_2)_z = 0 \qquad \ldots (2.12)$$

$$p_2 = A\gamma\rho_2 \qquad \ldots (2.13)$$

$$T_2 = (\gamma - 1)\rho_2. \qquad \ldots (2.14)$$

Third order : $$v_{3z} = -\frac{1}{r} u_2 (rv_1)_r. \qquad \ldots (2.15)$$

Fourth order : $$u_{4z}+A\gamma\rho_{4r} = -\tfrac{1}{2}[u_2^2+A\gamma(\gamma-2)\rho_2^2]_r - w_2u_{2z}+\frac{2}{r}v_1v_3 \quad \ldots (2.16)$$

$$(w_4+A\gamma\rho_4)_z = -\tfrac{1}{2}[w_2^2+A\gamma(\gamma-2)\rho_2^2]_z - u_2w_{2r} \quad \ldots (2.17)$$

$$\frac{1}{r}(ru_4)_r+(w_4+\rho_4)_z = -\rho_2\left[\frac{1}{r}(ru_2)_z+w_{2z}\right]-u_2\rho_{2r}-w_2\rho_{2z} \quad \ldots (2.18)$$

$$p_4 = A\gamma\left(\rho_4+\frac{\gamma-2}{2}\rho_2^2\right) \quad \ldots (2.19)$$

$$T_4 = (\gamma-1)\left(\rho_4+\frac{\gamma-2}{2}\rho_2^2\right). \quad \ldots (2.20)$$

We have adopted the method of Laplace transforms to solve the above equations. The solution of the fourth order perturbation equations involved triple series. We have therefore decided to work only up to this order but for the validity of the expansion of flow variables only up to this order we restrict the value of ϵ to such a level where the contribution of the fourth order perturbation terms is one order lower than the earlier terms.

Boundary Conditions

At the entry section, $z = 0$

$$u = 0, \quad v = \epsilon r, \quad w = 1, \quad \rho = 1 \qquad \text{for } 0 \leqslant r \leqslant 1. \quad \ldots (2.21)$$

As $z \to \infty$, the flow variables remain bounded

$$\text{at } r = 0, \quad u = 0, \quad v = 0 \quad \ldots (2.22)$$

and $$w_r = \rho_r = T_r = 0 \text{ (symmetry condition about } r = 0) \quad \ldots (2.23)$$

$$\text{at } r = 1, \quad u = 0. \quad \ldots (2.24)$$

We can easily show that the various order perturbations satisfy the following boundary conditions :

$$\left.\begin{aligned} &u_2 = w_2 = \rho_2 = 0 \text{ at } z = 0 \text{ for } 0 \leqslant r \leqslant 1 \\ &u_2 = 0,\ w_{2z} = \rho_{2z} = T_{2z} = 0 \text{ at } r = 0 \text{ for } z \geqslant 0 \\ &u_2 = 0 \text{ at } r = 1 \text{ for } z \geqslant 0 \end{aligned}\right\} \quad \ldots (2.25)$$

$$v_3 = 0 \text{ at } z = 0 \text{ for } 0 \leqslant r \leqslant 1 \quad \ldots (2.26)$$

and

$$\left.\begin{aligned} &u_4 = \rho_4 = u_4 = 0 \text{ at } z = 0 \text{ for } 0 \leqslant r \leqslant 1 \\ &u_4 = w_{4r} = \rho_{4r} = T_{4r} = 0 \text{ at } r = 0 \text{ for } z \geqslant 0 \\ &u_4 = 0 \text{ at } r = 1 \text{ for } z \geqslant 0. \end{aligned}\right\} \quad \ldots (2.27)$$

3. Solution for Various Order Perturbations

First Order Equation

From (2.9), we have

$$v_1 = r \text{ for } z \geqslant 0 \qquad \ldots (3.1)$$

Second Order Equations

In view of the boundary conditions in (2.25) the Laplace transforms $\bar{u}_2$, $\bar{w}_2$, $\bar{\rho}_2$, of u_2, w_2, ρ_2 are determined by

$$p\bar{u}_2 + A\gamma\bar{\rho}_{2r} = \frac{1}{p}\, r, \text{ on using (3.1)} \qquad \ldots (3.2)$$

$$p(\bar{w}_2 + A\gamma\bar{\rho}_2) = 0 \qquad \ldots (3.3)$$

$$\frac{1}{r}\,(r\bar{u}_2)_r + p(\bar{w}_2 + \bar{\rho}_2) = 0 \qquad \ldots (3.4)$$

where

$$\bar{x}(r, p) = \int_0^\infty e^{-pz} x(r, z)\, dz. \qquad \ldots (3.5)$$

We note that in writing (3.2)–(3.4) we have assumed that $u_2 e^{-pz}$ etc. $\to 0$ as $z \to \infty$.

Eliminating $\bar{w}_2$, $\bar{\rho}_2$ among (3.2) to (3.4) we have

$$\bar{\rho}_{2rr} + \frac{1}{r}\,\bar{\rho}_{2r} - p^2\beta^2\bar{\rho}_2 = \frac{2(1+\beta^2)}{p} \qquad \ldots (3.6)$$

where

$$\beta^2 = \frac{1}{A\gamma} - 1 = \frac{M^2 - 2}{2}\,. \qquad \ldots (3.7)$$

The equation (3.6) is of Sturm-Liouville type with transition at the point $A\gamma = 1$. Consequently, the solution of the equation is of different form for $A\gamma \gtrless 1$. Erdelyi has applied Langer's method to obtain a continuous solution throughout the transition region for the Strum-Lioville equation. This method can be applied in the present case also but we are interested in the supersonic case as in Laven-Fejer investigation and we shall, therefore, assume that $M > \sqrt{2}$ i.e. $\beta > 0$ or $A\gamma < 1$.

The solution of the set of eqns. (3.2)–(3.4) for $\beta > 0$ is

$$\bar{u}_2 = \frac{r}{p^2} - \frac{I_1(\beta p r)}{p^2 I_1(\beta p)} \qquad \ldots (3.8)$$

$$\bar{\rho}_2 = (1+\beta^2)\left[-\frac{2}{\beta^2 p^3} + \frac{I_0(\beta p r)}{\beta p^2 I_1(\beta p)}\right] \qquad \ldots (3.9)$$

$$\bar{w}_2 = -\frac{1}{1+\beta^2}\,\bar{\rho}_2 \qquad \ldots (3.10)$$

where $I_0(z)$ and $I_1(z)$ are the associated Bessel functions of order zero and one respectively. On inversion, (3.8)–(3.10) yield

$$u_2(r, z) = \Sigma A_n S_n J_1(j_n r) \qquad \ldots (3.11)$$

$$\rho_2(r, z) = (1+\beta^2)\Big[\frac{1}{4}(2r^2-1)+\frac{1}{\beta}\Sigma A_n C_n J_0(j_n r)\Big] \qquad \ldots (3.12)$$

$$w_2(r, z) = -\frac{1}{1+\beta^2}\rho_2(r, z) \qquad \ldots (3.13)$$

where

$$A_n = -\frac{2\beta}{j_n^2 J_0(j_n)}, \quad S_n = \sin\Big[\frac{j_n z}{\beta}\Big], \quad C_n = \cos\Big[\frac{j_n z}{\beta}\Big] \qquad \ldots (3.14)$$

j_n are positive zeroes of the Bessel function $J_1(z)$ arranged in ascending order of magnitude and the summation extends from $n = 1$ to infinity.

Third Order Equation

The equation (2.15) in view of (3.1) reduces to

$$v_{3z} = -2u_2 \qquad \ldots (3.15)$$

which on integration under the boundary condition

$$v_3(r, 0) = 0$$

gives

$$v_3(r, z) = -\beta^2\frac{r(1-r^2)}{4}+\frac{\beta}{2}\Sigma\frac{A_n}{j_n}C_n J_1(j_n r). \qquad \ldots (3.16)$$

We note that $v_3(r, z) = 0$ at $r = 0$ for $z \geqslant 0$.

Fourth Order Equations

The equations (2.16)–(2.18) after using the second order equations and (3.13) reduce to

$$u_{4z}+\frac{1}{1+\beta^2}\rho_{4r} = -u_2u_{2r}-w_2w_{2z}+2v_3-(\gamma-2)(1+\beta^2)w_2w_{2r} \qquad \ldots (3.17)$$

$$\Big(w_4+\frac{1}{1+\beta^2}\rho_4\Big)_z = -u_2w_{2r}-\{1+(\gamma-2)(1+\beta^2)\}w_2w_{2z} \qquad \ldots (3.18)$$

$$\frac{1}{r}(ru_4)_r+(w_4+\rho_4)_z = (1+\beta^2)(2+\beta^2)w_2w_{2z}+(1+\beta^2)u_2u_{2r} \qquad \ldots (3.19)$$

which have to be solved under the conditions (2.27).

Integrating (3.18) with respect to z after using (3.13), (2.10) and (3.15) and the boundary conditions at $z = 0$, we get

$$w_4+\frac{1}{1+\beta^2}\rho_4 = -\tfrac{1}{2}[1+(\gamma-2)(1+\beta^2)]w_2^2-\tfrac{1}{2}u_2^2-\tfrac{1}{2}rv_3. \qquad \ldots (3.20)$$

Eliminating ρ_4 and w_4 among (3.17)–(3.20), we have

$$\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}-\beta^2\frac{\partial^2}{\partial z^2}\right]u_4 = F(r,z) \qquad \ldots (3.21)$$

where

$$F(r,z) = \Sigma\ A_nP_n(r)S_n+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_mA_nQ_{mn}(r)S_mC_n \qquad \ldots (3.22)$$

$$P_n(r) = \frac{1+\beta^2}{\beta^2}\left[-j_n(\lambda+\beta^2)rJ_0(j_nr)+\lambda j_n^2\frac{2r^2-1}{4}J_1(j_nr)+\frac{4\beta^4}{1+\beta^2}J_1(j_nr)\right] \qquad \ldots (3.23)$$

and

$$Q_{mn}(r) = \frac{1+\beta^2}{\beta}\left[\frac{(\gamma+1)(1+\beta^2)}{\beta^2}j_mj_nJ_0(j_mr)J_1(j_nr)\right.$$

$$\left.+\left[\frac{\lambda j_m^2}{\beta^2}+2j_n^2\right]J_1(j_mr)J_0(j_nr)-\frac{4j_n}{r}J_1(j_mr)J_1(j_nr)\right] \qquad \ldots (3.24)$$

with

$$\lambda = \gamma+1+(\gamma-1)\beta^2. \qquad \ldots (3.25)$$

Taking the Laplace transform of (3.21) and solving the resultant equation under the condition

$$\bar{u}_4(r,p) = 0 \qquad \text{at } r=0 \quad \text{and} \quad r=1 \qquad \ldots (3.26)$$

we get

$$\bar{u}_4(r,p) = I_1(\beta pr)\left[\frac{K_1(\beta p)}{I_1(\beta p)}\int_0^1 tI_1(\beta pt)\bar{F}(t,p)dt\right.$$

$$\left.-\int_r^1 tK_1(\beta pt)\bar{F}(t,p)dt\right]-K_1(\beta pr)\int_0^r tI_1(\beta pt)\bar{F}(t,p)dt \qquad \ldots (3.27)$$

where

$$\bar{F}(r,p) = \Sigma\ \frac{A_nP_n(r)\beta j_n}{p^2\beta^2+j_n^2}+\frac{\beta}{2}\ \Sigma\ \Sigma\ A_mA_nQ_{mn}(r)\left[\frac{j_m+j_n}{p^2\beta^2+(j_m+j_n)^2}\right.$$

$$\left.+\frac{j_m-j_n}{p^2\beta^2+(j_m-j_n)^2}\right] \qquad \ldots (3.28)$$

On inverting (3.27) and simplifying considerably, we have

$$u_4(r,z) = -2\sum_{n=1}^{\infty}\sum_{\substack{s=1\\(s\neq n)}}^{\infty}\frac{A_nj_nJ_1(j_sr)S_sI_1}{j_s(j_s^2-j_n^2)J_0^2(j_s)}+\sum_{n=1}^{\infty}\frac{A_n\left(\frac{z}{\beta}\right)J_1(j_nr)C_nI_2}{j_nJ_0^2(j_n)}$$

$$+\sum_{n=1}^{\infty}A_nS_nA_1(r)+\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\substack{s=1\\(s\neq m)}}^{\infty}A_mA_nA_2(r)S_s$$

$$+\tfrac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_mA_nA_3(r)\bar{S}_{m+n}+\tfrac{1}{2}\sum_{m>n=1}^{\infty}\sum^{\infty}A_mA_nA_4(r)\bar{S}_{m-n} \qquad \ldots (3.29)$$

where

$$\bar{S}_{m+n} = \sin\left\{\frac{j_m+j_n}{\beta}\, z\right\}, \quad \bar{S}_{m-n} = \sin\left\{\frac{j_m-j_n}{\beta}\, z\right\}.$$

$$A_1(r) = \frac{\pi}{2}\, J_1(j_n r)\, I_4 + \frac{\pi}{2}\, Y_1(j_n r)\, I_5 + \frac{J_1(j_n r)}{j_n J_0^2(j_n)}\, I_3$$

$$+\left[\frac{-3J_1(j_m r)}{j_n^2 J_0^2(j_n)} + \frac{r J_0(j_n r)}{j_n J_0^2(j_n)} - \frac{\pi}{2}\, \frac{Y_0(j_n)}{J_0(j_n)}\, J_1(j_n r)\right] I_2 \qquad \ldots (3.30)$$

$$A_2(r) = -\frac{1}{j_s J_0^2(j_s)}\left[\frac{j_m+j_n}{(j_m+j_n)^2-j_s^2} + \frac{j_m-j_n}{(j_m-j_n)^2-j_s^2}\right] J_1(j_s r) I_6 \qquad \ldots (3.31)$$

$$A_3(r) = \frac{\pi}{2}\left[-\frac{Y_1(j_m+j_n)}{J_1(j_m+j_n)} J_1(\overline{j_m+j_n}\, r) I_7 + J_1(\overline{j_m+j_n}\, r) I_8 + Y_1(\overline{j_m+j_n}\, r) I_9\right] \qquad \ldots (3.32)$$

$$A_4 = \frac{\pi}{2}\left[-\frac{Y_1(j_m-j_n)}{J_1(j_m-j_n)}\, J_1(\overline{j_m-j_n}\, r)\, I_{10} + J_1(\overline{j_m-j_n}\, r) I_{11} + Y_1(\overline{j_m-j_n}\, r) I_{12}\right] \qquad \ldots (3.33)$$

$$I_1 = \int_0^1 t J_1(j_s t) P_n(t)\, dt, \quad I_2 = \int_0^1 t J_1(j_n t) P_n(t)\, dt, \quad I_3 = \int_0^1 t J_0(j_n t) P_n(t)\, dt$$

$$I_4 = \int_r^1 t Y_1(j_n t) P_n(t)\, dt, \quad I_5 = \int_0^r t J_1(j_n t) P_n(t)\, dt, \quad I_6 = \int_0^1 t J_1(j_s t) Q_{mn}(t)\, dt$$

$$I_7 = \int_0^1 t J_1(\overline{j_m+j_n}\, t) Q_{mn}(t)\, dt, \qquad I_8 = \int_r^1 t Y_1(\overline{j_m+j_n}\, t) Q_{mn}(t)\, dt$$

$$I_9 = \int_0^r t J_1(\overline{j_m+j_n}\, t) Q_{mn}(t)\, dt, \qquad I_{10} = \int_r^1 t J_1(\overline{j_m-j_n}\, t) Q_{mn}(t)\, dt$$

$$I_{11} = \int_r^1 t Y_1(\overline{j_m-j_n}\, t) Q_{mn}(t)\, dt, \qquad I_{12} = \int_0^r t J_1(\overline{j_m-j_n}\, t) Q_{mn}(t)\, dt \qquad \ldots (3.34)$$

Substituting the value of $P_n(t)$ from (3.23) into I_n ($n = 1, 2, \ldots, 5$) and evaluating the integrals with the help of integrals evaluated in Appendix A, we have

$$I_1 = -\frac{1+\beta^2}{\beta^2}\left[\frac{\lambda+\beta^2}{j_n^2-j_s^2} - \frac{2\lambda j_n^2}{(j_n-j_s)^2}\right] j_n j_n J_0(j_s) J_0(j_n), \quad n \neq s \qquad \ldots (3.35)$$

$$I_2 = \frac{1+\beta^2}{\beta^2}\left[\frac{2\beta^4}{1+\beta^2} - \frac{\lambda j_n^2}{24}\right] J_0^2(j_n) \qquad \ldots (3.36)$$

$$I_3 = -\frac{(1+\beta^2) j_n J_0^2(j_n)}{12\beta^2}\,(3\lambda+2\beta^2) \qquad \ldots (3.37)$$

$$I_4 = \frac{1}{2\beta^2}(1+\beta^2)(\lambda+\beta^2)\Big[\frac{1-r^2}{\pi} + r^2 J_1(j_n r) Y_1(j_n r)\Big]$$

$$+\frac{1+\beta^2}{2\beta^2}\lambda j_n^2\Big[\frac{r^3}{2j_n}\{Y_1(j_n r)J_1'(j_n r)+J_1(j_n r)Y_1'(j_n r)\}-\frac{1}{2j_n}Y_1(j_n)J_0(j_n)$$

$$+\frac{1}{6}\{Y_0(j_n)J_0(j_n)-Y_0(j_n r)J_0(j_n r)r^4-J_1(j_n r)Y_1(j_n r)r^4\}$$

$$+\frac{1}{3j_n}\{Y_1(j_n)J_0(j_n)-r^3 Y_1(j_n r)J_0(j_n r)\}+\frac{r^3}{3j_n^2}J_1(j_n r)Y_1(j_n r)$$

$$+\frac{1}{3\pi j_n^2}(1-r^2)\Big]+\Big[\beta^3-\frac{(1+\beta^2)\lambda j_n}{16\beta^2}\Big][-Y_2(j_n)J_0(j_n)-J_2(j_n)Y_0(j_n)$$

$$-2r^2 Y_1(j_n r)J_1(j_n r)+Y_2(j_n r)J_0(j_n r)r^2+J_2(j_n r)Y_0(j_n r)r^2] \quad \ldots (3.38)$$

and

$$I_5 = \Big[2\beta^2-\frac{(1+\beta^2)\lambda j_n^2}{8\beta^2}\Big]\Big[\Big(r^2-\frac{1}{j_n^2}\Big)J_1^2(j_n r)+r^2 J_1'^2(j_n r)\Big]$$

$$+\frac{1+\beta^2}{12\beta^2}\lambda j_n^2 r^4[J_1^2(j_n r)+J_2^2(j_n r)]-\frac{1+\beta^2}{2\beta^2}(\lambda+\beta^2)r^2 J_1^2(j_n r). \quad \ldots (3.39)$$

After simplifying the integrals involving products of three Bessel functions occurring in I_n $(n = 6, \ldots, 12)$ with the help of the integrals evaluated in Appendix B and using the expression for $Q_{mn}(r)$ from (3.24) we have

$$I_6 = \frac{1+\beta^2}{\beta}\Big[\frac{\lambda j_m}{\beta^2}+\frac{1}{j_m}(j_s^2-j_n^2-j_m^2)\Big]\, I_{13} \quad \ldots (3.40)$$

$$I_7 = \frac{1+\beta^2}{\beta}\Big[\Big(\frac{\lambda j_m}{\beta^2}+2j_n\Big)(j_m+j_n)I_{14}$$

$$-\Big\{\frac{\lambda j_m}{\beta^2}+\frac{(j_m+j_n)^2}{j_m}\Big\}J_1(j_m+j_n)J_0(j_m)L_0(j_n)\Big] \quad \ldots (3.41)$$

$$I_8 = \frac{1+\beta^2}{\beta}\Big[\Big(\frac{\lambda j_m}{\beta^2}+2j_n\Big)(j_m+j_n)I_{15}+\Big\{\frac{\lambda j_m}{\beta^2}+\frac{(j_m+j_n)^2}{j_m}\Big\}$$

$$\times\{r Y_1(\overline{j_m+j_n}r)J_0(j_m r)J_0(j_n r)-Y_1(j_m+j_n)J_0(j_m)J_0(j_n)\}$$

$$-r\{2j_n J_1(j_n r)J_1(j_m r)Y_1(\overline{j_m+j_n}r)-(j_m+j_n)J_1(j_m r)J_0(j_n r)Y_0(\overline{j_m+j_n}r)$$

$$-\frac{j_n}{j_m}(j_m+j_n)J_1(j_n r)J_0(j_m r)Y_0(\overline{j_m+j_n}r)\Big] \quad \ldots (3.42)$$

$$I_9 = \frac{1+\beta^2}{\beta}\Big[\Big(\frac{\lambda j_m}{\beta^2}+2j_n\Big)(j_m+j_n)I_{16}+r\Big\{-\Big(\frac{\lambda j_m}{\beta^2}+\frac{(j_m+j_n)^2}{j_m}\Big)J_1(\overline{j_m+j_n}r)$$

$$\times J_0(j_m r)J_0(j_n r)+2j_n J_1(j_m r)J_1(j_n r)J_1(\overline{j_m+j_n}r)+(j_m+j_n)J_1(j_m r)$$

$$\times J_0(j_n r)J_0(\overline{j_m+j_n}r)+\frac{j_n}{j_m}(j_m+j_n)J_1(j_n r)J_0(j_m r)J_0(\overline{j_m+j_n}r)\Big\}\Big] \quad \ldots (3.43)$$

$$I_{10} = \frac{1+\beta^2}{\beta}\Big[\Big(\frac{\lambda j_m}{\beta^2}-2j_n\Big)(j_m-j_n)I_{17}-\Big\{\frac{\lambda j_m}{\beta^2}+\frac{(j_m-j_n)^2}{j_m}\Big\}$$

$$\times J_1(j_m-j_n)J_0(j_m)J_0(j_n)\Big] \quad \ldots (3.44)$$

$$I_{11} = \frac{1+\beta^2}{\beta}\Big[\Big(\frac{\lambda j_m}{\beta^2}-2j_n\Big)(j_m-j_n)I_{18}+\Big\{\frac{\lambda j_m}{\beta^2}-\frac{(j_m-j_n)^2}{j_m}\Big\}\{rY_1(\overline{j_m-j_n}r)$$

$$\times J_0(j_m r)J_0(j_n r)-Y_1(j_m-j_n)J_0(j_m)J_0(j_n)\}-r\Big\{\frac{j_n}{j_m}(j_m-j_n)J_1(j_n r)$$

$$\times Y_0(\overline{j_m-j_n}r)+(j_m-j_n)J_1(j_m r)Y_0(j_n r)Y_0(\overline{j_m-j_n})+2j_n J_1(j_m r)$$

$$\times J_1(j_n r)Y_1(\overline{j_m-j_n}r)\Big\}\Big]. \quad \ldots (3.45)$$

and

$$I_{12} = \frac{1+\beta^2}{\beta}\Big[\Big(\frac{\lambda j_m}{\beta^2}-2j_n\Big)(j_m-j_n)I_{19}+r\Big\{-\Big(\frac{\lambda j_m}{\beta^2}+\frac{(j_m-j_n)^2}{j_m}\Big)J_1(\overline{j_m-j_n}r)$$

$$\times J_0(j_m r)J_0(j_n r)+(j_m-j_n)J_0(\overline{j_m-j_n}r)J_1(j_m r)J_0(j_n r)+\frac{j_n}{j_m}(j_m-j_n)$$

$$\times J_1(j_n r)J_0(j_m r)J_0(\overline{j_m-j_n}r)+2j_n J_1(\overline{j_m-j_n}r)J_1(j_m r)J_1(j_n r)\Big\}\Big] \quad \ldots (3.46)$$

where

$$I_{13} = \int_0^1 T(t)J_0(j_s t)dt$$

$$I_{14} = \int_0^1 T(t)J_0(\overline{j_m+j_n}t)dt$$

$$I_{15} = \int_r^1 T(t)Y_0(\overline{j_m+j_n}t)dt$$

$$I_{16} = \int_0^r T(t)J_0(\overline{j_m+j_n}t)dt$$

$$I_{17} = \int_0^1 T(t)J_0(\overline{j_m-j_n}t)dt$$

$$I_{18} = \int_r^1 T(t)Y_0(\overline{j_m-j_n}t)dt$$

$$I_{19} = \int_0^r T(t)J_0(\overline{j_m-j_n}t(dt)$$

with

$$T(t) = t\, J_0(j_m t) J_0(j_n t).$$

One of us (P.L.B.) has prepared detailed tables* for the above integrals for all admissible combinations of the zeros, j_m, j_n, j_s for values 1 to 5 of m, n and s using 0·01 as the interval for r, r lying between 0 and 1.

Using the above values of I_n ($n = 1, \ldots, 12$) the expression (3.29) for $u_4(r, z)$ simplifies to

$$\frac{\beta u_4(r,z)}{2(1+\beta^2)} = \left[\frac{\beta^3}{8(1+\beta^2)} r(1-r^2) + \frac{\beta v_3(r,z)}{2(1+\beta^2)}\right] z + \beta u_2(r,z)\left[\tfrac{1}{2} w_2(r,z) + \frac{\lambda(1-2r^2)}{48\beta^2} + \frac{3r^2-2}{24}\right] + \sum_{n=1}^{\infty} K_n j_n \left[\frac{\lambda z}{24\beta} C_n J_1(j_n r) + S_n\left\{-\frac{\lambda}{12} r(1-r^2) J_0(j_n r) + \frac{2\beta^4 J_1(j_n r)}{(1+\beta^2) j_n^3}\right\}\right] - 2 \sum_{n=1}^{\infty} \sum_{\substack{s=1 \\ (s \neq n)}}^{\infty} K_s j_s^2 J_1(j_s r) S_s \left\{\frac{\beta^2}{(j_n - j_s)^2} \times \right.$$

$$\left. \times \frac{\lambda(j_n^2 + j_s^2)}{(j_n^2 - j_s^2)^2}\right\} - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{\substack{s=1 \\ (s \neq m,\, n)}}^{\infty} \frac{2 K_m K_n K_s^2 j_s^4}{j_m} I_{13} J_1(j_s r) S_s \{(\lambda - \beta^2) j_m^2 + \beta^2 (j_s^2 - j_n^2)\} \cdot$$

$$\left\{\frac{j_m + j_n}{(j_m + j_n)^2 - j_s^2} + \frac{j_m - j_n}{(j_m - j_n)^2 - j_s^2}\right. + \sum_{m=1}^{\infty} \sum_{m=1}^{\infty} K_m K_n \left\{-\frac{2 j_n \beta^2}{j_m} \cdot \right.$$

$$\times J_0(j_m r) J_1(j_n r) S_m C_n + (j_m + j_n) \bar{S}_{m+n} (\lambda j_m + 2\beta^2 j_n) \bar{A}_3(r) + (j_m - j_n)$$

$$\left. \times (\lambda j_m - 2\beta^2 j_n) \bar{S}_{m-n} \bar{A}_4(r)\right\}. \qquad (3.47)$$

where

$$K_n = \frac{1}{j_n^2 J_0(j_n)} \qquad \ldots (3.48)$$

$$\bar{A}_3(r) = \frac{\pi}{2}\left[-\frac{Y_1(j_m + j_n)}{J_1(j_m + j_n)} J_1(\overline{j_m + j_n} r) I_{14} + J_1(\overline{j_m + j_n} r) I_{15} + Y_1(\overline{j_m + j_n} r) I_{16}\right]. \qquad \ldots (3.49)$$

and

$$\bar{A}_4(r) = \frac{\pi}{2}\left[\frac{Y_1(j_m - j_n)}{J_1(j_m - j_n)} J_1(\overline{j_m - j_n} r) I_{17} + J_1(\overline{j_m - j_n} r) I_{18} + Y_1(\overline{j_m - j_n} r) I_{19}\right]. \qquad \ldots (3.50)$$

* To be Published.

Now combining (2.16)–(2.18) suitably, we get the equation determining ρ_4 in terms of u_4

$$\frac{\beta^2}{1+\beta^2}\rho_{4z} = -\frac{1}{r}(ur_4)_r + \frac{1+(1+\beta^2)(\gamma+\beta^2)}{2(1+\beta^2)^2}\frac{\partial}{\partial z}(\rho_2^2) + \frac{2+\beta^2}{\partial z}\frac{\partial}{2}(u_2^2) + \frac{2+\beta^2}{2}rv_{3z} \quad \ldots (3.51)$$

which on integration with respect to z and using the boundary conditions $u_2 = \rho_2 = v_3 = 0$ at $z = 0$ yields

$$\frac{\beta^2}{2(1+\beta^2)^2}\rho_4(r,z) = \frac{(1+\beta^2)(\gamma+1)+\beta^4}{4(1+\beta^2)^3}\rho_2^2 + \frac{(2+\beta^2)u_2^2}{4(1+\beta^2)} + \left[\frac{1}{8} + \frac{1}{4(1+\beta^2)} - \frac{\lambda}{24\beta^2}\right]rv_3 + \left[\frac{\lambda(3-5r^2)}{12} + \frac{\beta^2(1-3r^2)}{12}\right]\frac{\rho_2}{2(1+\beta^2)}$$

$$+ \sum_{n=1}^{\infty}\sum_{\substack{s=1\\(s\neq n)}}^{\infty} 4K_s j_s^2 J_0(j_s r)\sin^2\left(\frac{j_s z}{2\beta}\right)\left\{\frac{\beta^2}{(j_n^2-j_s^2)^2} - \frac{\lambda(j_n^2+j_s^2)}{(j_n^2-j_s^2)^3}\right\}$$

$$+ \sum_{n=1}^{\infty} K_n\left(\frac{j_n z}{\beta}\right)S_n J_0(j_n r)\frac{2\beta^4}{1+\beta^2}\frac{1}{j_n^2} - \frac{\lambda}{24} - \sum_{n=1}^{\infty} 2K_n\sin^2\left(\frac{j_n z}{2\beta}\right)\left\{\frac{4\beta^4}{1+\beta^2}\right.$$

$$\left. + \frac{J_0(j_n r)}{j_n^2} + \frac{\lambda j_n r(1-r^2)J_1(j_n r)}{12}\right\} - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} 2K_m K_n S_m^2\left\{\frac{(j_m+j_n)z}{2\beta}\right\}\times$$

$$\times\left[\left\{(j_m+j_n)B_3(r) - \frac{J_0(j_m r)J_0(j_n r)}{j_m+j_n}\right\}(\lambda j_m + 2\beta^2 j_n) - \frac{\beta^2}{j_m+j_n}\left\{\frac{j_m^2+j_n^2}{j_m}\right.\right.$$

$$\left.\left.\times J_0(j_m r)J_0(j_n r) - 2j_n J_1(j_m r)J_1(j_n r)\right\}\right] - \sum_{m=1}^{\infty}\sum_{\substack{n=1\\(n\neq m)}}^{\infty} SK_m K_n\sin^2\left\{\frac{(j_m-j_n)z}{2\beta}\right.$$

$$\times\left[\left\{(j_m-j_n)B_4(r) - \frac{J_0(j_m r)J_0(j_m r)}{j_m-j_n}\right\}(\lambda j_m - 2\beta^2 j_n) - \frac{\beta^2}{j_m-j_n}\left\{\frac{j_m^2+j_n^2}{j_m}\right.\right.$$

$$\left.\left.\times J_0(j_m r)J_0(j_n r) - 2j_n J_1(j_m r)J_1(j_n r)\right\}\right] + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\substack{s=1\\(s\neq m,\ n)}}^{\infty}\frac{4j_s^2}{j_m}K_m K_n K_s^2$$

$$\times J_0(j_s r)\sin^2\left(\frac{j_s z}{2\beta}\right)I_{13}\{(\lambda-\beta^2)j_m^2 + \beta^2(j_s^2-j_n^2)\}\left\{\frac{j_m+j_n}{(j_m+j_n)^2-j_s^2} + \frac{j_m-j_n}{(j_m-j_n)^2-j_s^2}\right\}$$

where ... (3.52)

$$B_3(r) = \frac{\pi}{2}\left[-\frac{Y_1(j_m+j_n)}{J_1(j_m+j_n)}J_0(\overline{j_m+j_n}\,r)I_{14} + J_0(\overline{j_m+j_n}\,r)I_{15} + Y_0(\overline{j_m+j_n}\,r)I_{16}\right] \quad \ldots (3.53)$$

and

$$B_4(r) = \frac{\pi}{2}\left[-\frac{Y_1(j_m-j_n)}{J_1(j_m-j_n)}\, J_0(\overline{j_m-j_n}r)I_{17}+J_0(\overline{j_m-j_n}r)I_{18}+Y_0(\overline{j_m-j_n}r)I_{19}\right]. \quad \ldots (3.54)$$

Having determined ρ_4, (3.20) gives the value of w_4.

To the approximation to which we have worked, the ratio of mass flux at any cross-section z to that at $z = 0$:

$$\frac{Q}{Q_0} = \frac{\int_0^1 2\pi r dr \rho w}{\int_0^1 2\pi r dr \rho_0 w_0} \quad \ldots (3.55)$$

is given by

$$\frac{Q}{Q_0} = 1+2\epsilon^2 \int_0^1 \left(\frac{T_2}{\gamma-1}+w_2\right) r dr + 2\epsilon^4 \int_0^1 r dr \left[\frac{T_4}{\gamma-1}+w_4+\frac{w_2 T_2}{\gamma-1}+\frac{1}{2}\right.$$

$$\left.\times\frac{(2-r)}{(\gamma-1)^2}\, T_2^2\right] \quad \ldots (3.56)$$

which can be easily computed if ϵ is prescribed and values of T_2, w_2, T_4, w_4 are known.

4. Discussion of the Results

In the numerical work, we have taken $M = 2$, i.e., $\beta = 1$ and $\gamma = 1{\cdot}4$ corresponding to the dry air. Table I gives the values of v_1, v_3, u_2, u_4, w_2, w_4, T_2 and T_4 for a selected set of values of z, namely $z = 0{\cdot}5$, $1{\cdot}0$, $5{\cdot}0$, 10, 15, 20, 25, 30, 35 and 40 and for some selected values of r, namely $r = 0{\cdot}1$, $0{\cdot}2$, ..., $1{\cdot}0$. for each of these values of z. In obtaining these values we have summed up each series up to 10 terms so that the double series has been summed up upto 10×10 terms and so on. Figures 1, 2, 3 give respectively the temperature, axial velocity and radial velocity profiles at the cross-sections indicated above for $\epsilon = 0{\cdot}03$, $0{\cdot}05$ and $0{\cdot}08$. The azimuthal velocity profiles show insignificant deviations from the initial solid-body rotation and consequently we have not plotted them. We have restricted the values of ϵ up to $0{\cdot}08$ in order to ensure that the fourth order perturbations give contributions to flow variables of at least one order smaller than the previous, though it is surmized that the present investigation is valid even for slightly higher values of ϵ than those considered here. Similarly, the choice of the largest value of z is restricted to $z = 40$ as in experiments the tube lengths usually do not exceed this value. Table 2 gives the excess $\Delta T = T_1 - T_0$ of the wall temperature over the axial temperature multiplied by 300 at the

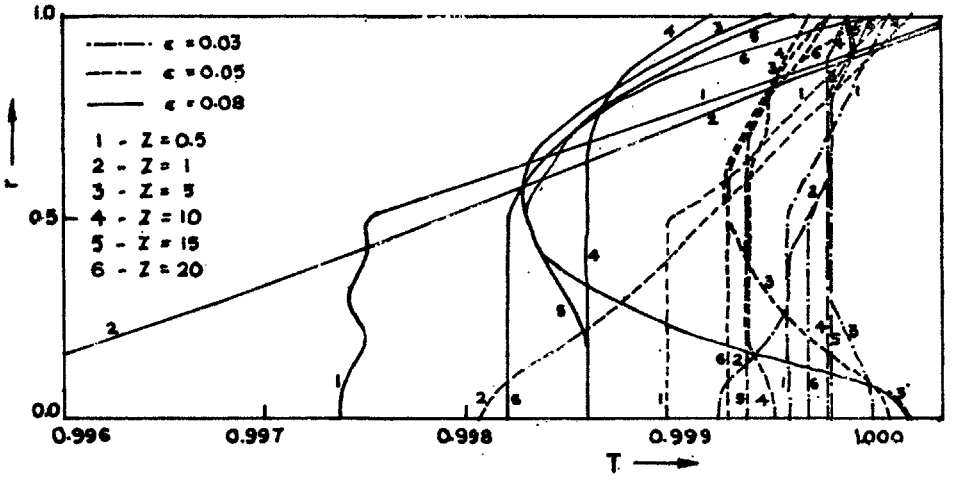

FIG. 1 (a).

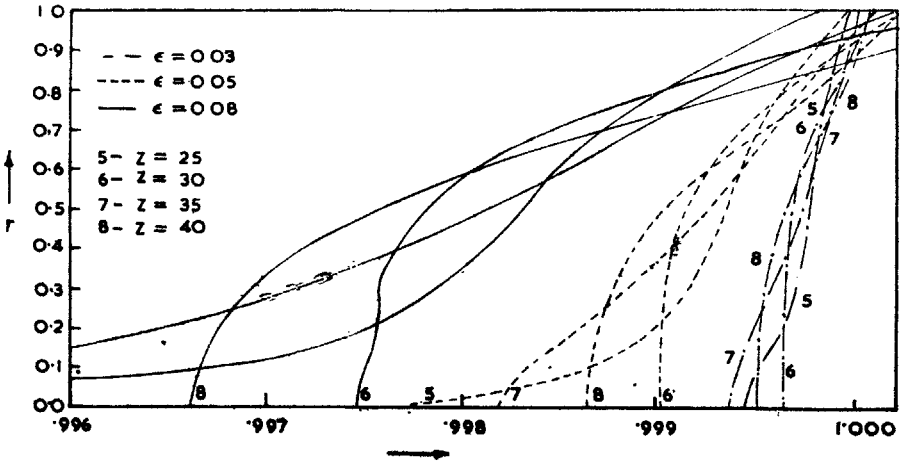

FIG. 1 (b).

selected cross-sections. We have multiplied ΔT by 300 to give in degrees centrigrade the actual excess in degrees centigrade if the entry temperature is 37°C.

From Figs. 1–3 and the Table II we can draw the following conclusions :

(i) The temperature profile shows at first significant deviation from the uniform entry temperature profile $T = 1$ so much so that the maximum

positive excess is obtained at $z = 1$. Then this excess shows reversal in sign and at $z = 5$, the axial temperature is actually higher than the wall temperature. After this, ΔT remains positive but shows oscillations. This behaviour

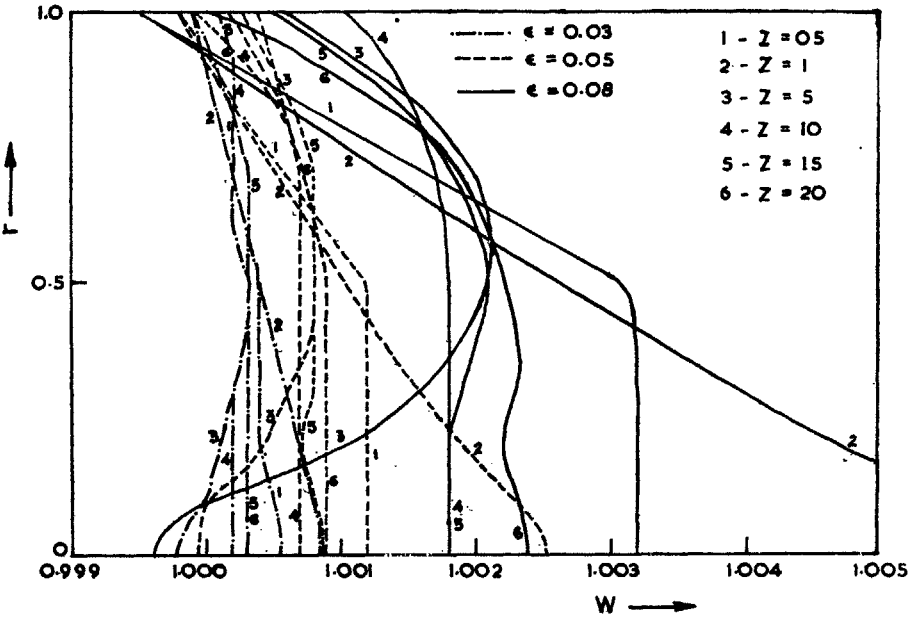

FIG. 2 (a).

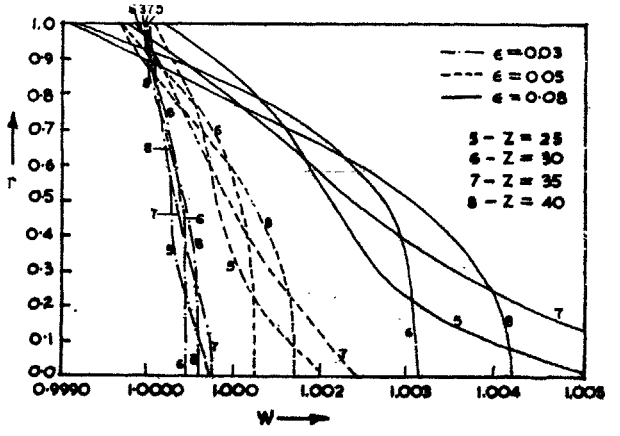

FIG. 2 (b).

of ΔT becomes more and more pronounced as ϵ increases, but even for $\epsilon = 0{\cdot}08$ the maximum excess of wall temperature over the axial temperature occurring near $z = 1$ is hardly a degree and half, but it is clear from the Table II that for larger values of ϵ, 300 ΔT may attain much bigger values.

(ii) The values of u remain small but the u-profiles show considerable variation.

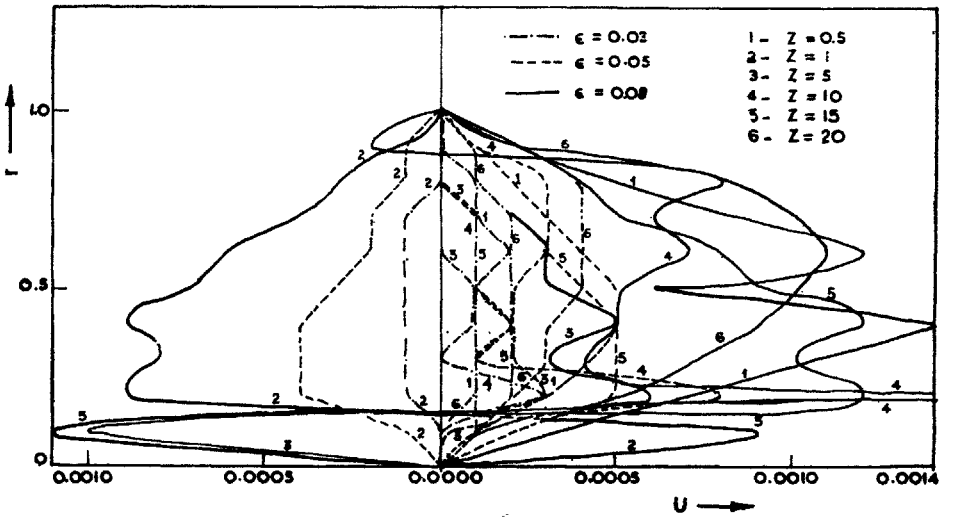

FIG. 3 (a).

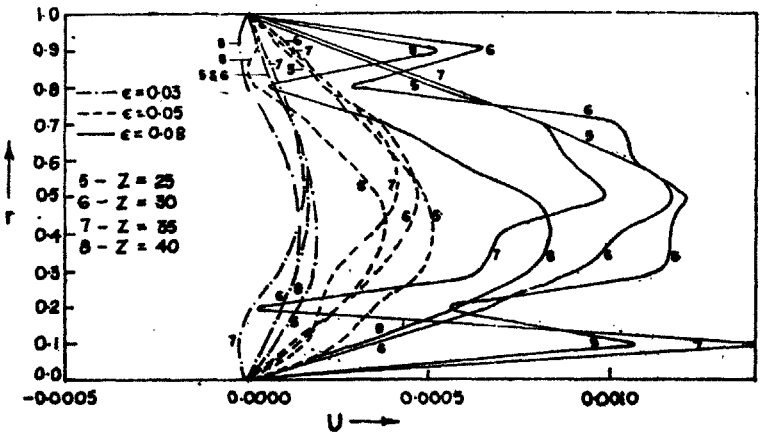

FIG. 3 (b).

Table I

z	r	v_1	v_3	u_2	u_4	w_2	w_4	T_2	T_4
0·5	0·1	0·1000	−0·0248	0·0517	−1·2451	0·5013	−0·5699	−0·4011	0·3553
0·5	0·2	0·2000	−0·0485	0·0991	2·1049	0·4989	−0·7727	−0·3991	0·5185
0·5	0·3	0·3000	−0·0700	0·1502	1·4052	0·5008	−0·5039	−0·4006	0·3022
0·5	0·4	0·4000	−0·0880	0·2012	2·0688	0·5003	−0·5000	−0·4003	0·2978
0·5	0·5	0·5000	−0·1016	0·2362	−21·0581	0·4871	−0·3295	−0·3897	0·1667
0·5	0·6	0·6000	−0·1063	0·1737	2·3891	0·3672	−0·3559	−0·2937	0·2443
0·5	0·7	0·7000	−0·0989	0·1149	1·7791	0·2478	−0·1623	−0·1982	0·1277
0·5	0·8	0·8000	−0·0791	0·0699	1·0412	0·1356	−0·1190	−0·1076	0·1113
0·5	0·9	0·9000	−0·0464	0·0325	0·4840	0·0311	−0·0770	−0·0249	0·0775
0·5	1·0	1·0000	0·0000	0·0000	−0·0181	−0·0725	−0·1839	0·0580	0·1450
1·0	0·1	0·1000	−0·0367	−0·1607	47·6046	0·9114	−0·6834	−0·7291	0·2056
1·0	0·2	0·2000	−0·0661	−0·1635	2·5719	0·7468	−0·8820	−0·5975	0·4771
1·0	0·3	0·3000	−0·0893	−0·1620	6·4812	0·6251	−0·6560	−0·5001	0·3687
1·0	0·4	0·4000	−0·1053	−0·1413	−0·4220	0·5131	−0·5033	−0·4104	0·3062
1·0	0·5	0·5000	−0·1136	−0·1226	1·8967	0·4084	−0·4097	−0·3267	0·2778
1·0	0·6	0·6000	−0·1130	−0·0965	−0·7145	0·3091	−0·2594	−0·2473	0·1927
1·0	0·7	0·7000	−0·1024	−0·0728	0·6222	0·2098	−0·1828	−0·1678	0·1552
1·0	0·8	0·8000	−0·0809	−0·0468	−0·3666	0·1136	−0·0072	−0·0909	0·0256
1·0	0·9	0·9000	−0·0471	−0·0230	0·1311	0·0145	−0·0104	−0·0116	0·0250
1·0	1·0	1·0000	0·0000	0·0000	−0·0181	−0·0833	−0·1196	0·0667	0·0929
5·0	0·1	0·1000	−0·0106	0·1218	−43·4613	−0·0040	−0·5546	0·0032	0·4382
5·0	0·2	0·2000	−0·0261	0·1041	−0·4619	0·1516	0·3064	−0·1213	−0·2566
5·0	0·3	0·3000	−0·0426	0·0927	−4·7534	0·2428	0·3813	−0·1942	−0·3270
5·0	0·4	0·4000	−0·0578	0·0709	1·4829	0·3022	0·2160	−0·2417	−0·2021
5·0	0·5	0·5000	−0·0692	0·0569	−1·0429	0·3313	0·0620	−0·2651	−0·0810
5·0	0·6	0·6000	−0·0750	0·0404	1·2118	0·3323	−0·0672	−0·2658	0·0269
5·0	0·7	0·7000	−0·0728	0·0291	−0·2509	0·3104	−0·0804	−0·2483	0·0458
5·0	0·8	0·8000	−0·0610	0·0174	0·5400	0·2627	−0·2499	−0·2101	0·1917
5·0	0·9	0·9000	−0·0373	0·0085	0·0540	0·1953	−0·3254	−0·1563	0·2585
5·0	1·0	1·0000	0·0000	0·0000	−0·0181	0·1042	−0·4922	−0·0834	0·3894
10·0	0·1	0·1000	−0·0192	0·0149	0·5889	0·2744	−0·0711	−0·2195	0·0266
10·0	0·2	0·2000	−0·0373	0·3286	0·8240	0·2763	0·1297	−0·2210	−0·1318
10·0	0·3	0·3000	−0·0531	0·0493	0·1294	0·2779	−0·1714	−0·2223	0·1116
10·0	0·4	0·4000	−0·0658	0·0662	2·1059	0·2800	−0·0689	−0·2240	0·0325
10·0	0·5	0·5000	−0·0739	0·0847	−0·5882	0·2801	−0·5553	−0·2241	0·4248
10·0	0·6	0·6000	−0·0763	0·0985	2·7839	0·2778	−0·4849	−0·2222	0·3715
10·0	0·7	0·7000	−0·0717	0·1135	−3·5055	0·2691	−0·8054	−0·2153	0·6303
10·0	0·8	0·8000	−0·0587	0·1111	2·6138	0·2508	−0·5645	−0·2006	0·4403
10·0	0·9	0·9000	−0·0355	0·1000	−19·7881	0·2154	−0·1723	−0·1723	0·1280
10·0	1·0	1·0000	0·0000	0·0000	−0·0181	0·1459	2·1053	−0·1167	−1·6928
15·0	0·1	0·1000	−0·0213	0·2149	−61·4119	0·3285	−8·1522	−0·2628	6·4609
15·0	0·2	0·2000	−0·0392	0·2250	−6·5470	0·2932	−1·7529	−0·2346	1·3508
15·0	0·3	0·3000	−0·0554	0·2176	−8·8403	0·3036	−0·2624	−0·2428	0·1608
15·0	0·4	0·4000	−0·0688	0·1843	0·2837	0·3163	0·3847	−0·2531	−0·3504
15·0	0·5	0·5000	−0·0781	0·1537	−1·8604	0·3215	0·4972	−0·2572	−0·4329
15·0	0·6	0·6000	−0·0816	0·1168	1·3612	0·3131	0·2663	−0·2505	−0·2382
15·0	0·7	0·7000	−0·0775	0·0853	−0·2291	0·2848	0·2341	−0·2279	−0·2009
15·0	0·8	0·8000	−0·0638	0·0534	0·7729	0·2381	−0·0503	−0·1905	0·0369
15·0	0·9	0·9000	−0·0387	0·0258	0·0053	0·1683	−0·0450	−0·1346	0·0383
15·0	1·0	1·0000	0·0000	0·0000	−0·0181	0·0796	−0·3837	−0·0637	0·3044

Table I (Contd.)

z	r	v_1	v_3	u_2	u_4	w_2	w_4	T_2	T_4
20·0	0·1	0·1000	−0·0212	0·0352	0·3339	0·3586	−0·8860	−0·2869	0·6577
20·0	0·2	0·2000	−0·0414	0·0703	0·0680	0·3571	−1·0147	−0·2857	0·7621
20·0	0·3	0·3000	−0·0594	0·1033	0·4829	0·3635	−1·0552	−0·2908	0·7942
20·0	0·4	0·4000	−0·0740	0·1332	0·8114	0·3603	−1·1824	−0·2882	0·8987
20·0	0·5	0·5000	−0·0841	0·1579	−0·3026	0·3571	−1·5440	−0·2857	1·1914
20·0	0·6	0·6000	−0·0879	0·1725	−0·4259	0·3359	−1·4614	−0·2688	1·1332
20·0	0·7	0·7000	−0·0839	0·1738	−3·2848	0·3008	−1·1909	−0·2407	0·9280
20·0	0·8	0·8000	−0·0696	0·1514	−3·8667	0·2325	−0·3204	−0·1860	0·2478
20·0	0·9	0·9000	−0·0428	0·0996	−10·6942	0·1244	1·2596	−0·0995	−1·0024
20·0	1·0	1·0000	0·0000	0·0000	−0·0181	−0·0211	3·1439	0·0169	−2·5153
25·0	0·1	0·1000	−0·0315	0·0664	15·6455	0·6931	−6·4077	−0·5345	4·9335
25·0	0·2	0·2000	−0·0550	0·1685	−1265·95	0·5193	−5·7603	−0·4155	4·4934
25·0	0·3	0·3000	−0·0730	0·2083	−4·4147	0·4296	−2·7659	−0·3436	2·1303
25·0	0·4	0·4000	−0·0857	0·2189	−5·7230	0·3711	−0·9291	−0·2969	0·6827
25·0	0·5	0·5000	−0·0930	0·2004	−1·6309	0·3295	−0·0691	−0·2636	0·0144
25·0	0·6	0·6000	−0·0936	0·1701	−1·9475	0·2924	0·5200	−0·2339	−0·4393
25·0	0·7	0·7000	−0·0864	0·1292	−0·1821	0·2465	0·4384	−0·1972	−0·3575
25·0	0·8	0·8000	−0·0696	0·0862	−0·4668	0·1910	0·6968	−0·1528	−0·5527
25·0	0·9	0·9000	−0·0414	0·0418	0·0353	0·1163	0·4256	−0·0930	−0·3317
25·0	1·0	1·0000	0·0000	0·0000	−0·0181	0·0269	0·4605	−0·0215	−0·3687
30·0	0·1	0·1000	−0·0248	0·0549	0·5083	0·5038	−2·8273	−0·4030	2·1601
30·0	0·2	0·2000	−0·0136	0·1036	0·7641	0·4986	−3·7915	−0·3989	2·9339
30·0	0·3	0·3000	−0·0697	0·1468	0·3554	0·4906	−2·5026	−0·3924	1·9055
30·0	0·4	0·4000	−0·0870	0·1784	−2·8171	0·4702	−2·9095	−0·3762	2·2403
30·0	0·5	0·5000	−0·0987	0·1921	−1·3040	0·4362	−1·5783	−0·3490	1·1987
30·0	0·6	0·6000	−0·1029	0·1879	−5·9331	0·3778	−1·2467	−0·3022	0·9509
30·0	0·7	0·7000	−0·9975	0·1538	−1·6643	0·2921	0·8720	−0·2337	−0·7139
30·0	0·8	0·8000	−0·0799	0·1018	−8·9872	0·1725	1·9218	−0·1380	−1·5279
30·0	0·9	0·9000	−0·0480	0·0216	12·5817	0·0246	2·8271	−0·0194	−2·2448
30·0	1·0	1·0000	0·0000	0·0000	−0·0181	−0·1164	−2·9975	0·0931	2·3925
35·0	0·1	0·1000	−0·0329	−0·0936	63·6946	0·8179	6·9173	−0·6543	−5·8037
35·0	0·2	0·2000	−0·0626	0·0139	−1·8121	0·7169	−5·9668	−0·5735	4·5721
35·0	0·3	0·3000	−0·0855	0·0933	1·2771	0·5896	−5·3922	−0·4717	4·1814
35·0	0·4	0·4000	−0·1009	0·1542	−7·2170	0·4761	≠3·9168	−0·3809	3·0493
35·0	0·5	0·5000	−0·1085	0·1726	−2·8227	0·3792	−2·2481	−0·3033	1·7507
35·0	0·6	0·6000	−0·1077	0·1687	−4·7997	0·2936	−0·3881	−0·2349	0·2905
35·0	0·7	0·7000	−0·0977	0·1390	−1·7436	0·2146	0·3413	−0·1717	−0·2719
35·0	0·8	0·8000	−0·0773	0·0984	−2·0019	0·1334	1·4538	−0·1067	−1·4935
35·0	0·9	0·9000	−0·0452	0·0491	−0·5202	0·0477	1·3295	−0·0382	−1·4922
35·0	1·0	1·0000	0·0000	0·0000	−0·0181	−0·0480	1·7351	0·0384	−1·3874
40·0	0·1	0·1000	−0·0296	0·0582	−1·2677	0·6874	−6·1005	−0·5499	4·6912
40·0	0·2	0·2000	−0·0575	0·1081	−2·4746	0·6682	−5·9117	−0·5346	4·5506
40·0	0·3	0·3000	−0·0821	0·1448	−3·1815	0·6239	−4·1118	−0·4991	3·1351
40·0	0·4	0·4000	−0·1012	0·1616	−5·0976	0·5649	−2·8642	−0·4519	2·1694
40·0	0·5	0·5000	−0·1131	0·1536	−4·4901	0·4757	−0·3992	−0·3806	0·2419
40·0	0·6	0·6000	−0·1155	0·1223	−5·8934	0·3673	1·3690	−0·2938	−1·1274
40·0	0·7	0·7000	−0·1065	0·0684	−1·5788	0·2328	3·4664	−0·1862	−2·7669
40·0	0·8	0·8000	−0·0845	0·0112	−0·5582	0·0940	3·4811	−0·0752	−2·7614
40·0	0·9	0·9000	−0·0486	−0·0305	15·0068	0·0282	0·3485	−0·0225	−0·2619
40·0	1·0	1·0000	0·0000	0·0000	−0·0181	−0·1035	−6·4711	0·0828	5·1726

Table II

	z	0·5	1·0	5·0	10	15	20	25	30	35	40	ϵ
300 Δt		0·15	0·24	−0·03	0·03	0·03	0·09	0·15	0·15	0·18	0·16	0·03
		0·33	0·60	−0·06	0·06	0·12	0·21	03·9	0·36	0·54	0·48	0·05
		0·90	1·56	−0·15	0·18	0·03	0·54	0·99	0·96	1·50	1·20	0·08

(iii) The w-profiles show tendency just opposite to the corresponding T-profiles over all cross-sections. This can be understood as follows. If we determine the characteristics and the corresponding compatibility conditions for the set of equations (2.1) to (2.4) after substituting for ρ in terms of T from (2.6), we can easily show that along each of the family of characteristics defined by

$$\frac{dr}{dz} = \frac{u}{w} \qquad \ldots \quad (4.1)$$

$$rv = \text{constant} \qquad \ldots \quad (4.2)$$

and

$$\frac{2}{M^2(r-1)}\, T + \tfrac{1}{2}(u^2+v^2+w^2) = \text{constant}. \qquad \ldots \quad (4.3)$$

since at the axis $r = 0$, $u = v = 0$, the axis is a characteristic of the family (4.1) and along it

$$\frac{2}{M^2(\gamma-1)}\, T + \tfrac{1}{2}w^2 = \frac{2}{M^2(\gamma-1)} + \tfrac{1}{2}. \qquad \ldots \quad (4.4)$$

Similarly, $r = 1$ is a characteristic of the family (4.1) along which $u = 0$, $v = \epsilon$ so that (4.3) again reduces to (4.4). Even in the general case, since $|u|$ is very small, the other members of the characteristic family are more or less parallel to the axis and along each characteristic the relation $v \simeq \epsilon r$ holds, so that now (4.4) holds approximately. The relation (4.4) explain the complementary behaviour of T- and w-profiles exhibited by Figs. 1 and 2.

(iv) If we denote by Q and Q_0 the rate of mass flow across the cross-sections at z and $z = 0$, we have approximately

$$\frac{Q}{Q_0} \simeq \quad 0{\cdot}9998,\ 0{\cdot}9994,\ 0{\cdot}9994 \text{ for } \epsilon = 0{\cdot}03,\ 0{\cdot}05,\ 0{\cdot}08$$

for the values of z recorded above. Thus the presence of swirl does produce in a supersonic axial flow a decrease in the mass flow rate as observed by Laven and Fejer though for small values of ϵ that we have considered here the decrease is extremely small.

Finally, we note that the occurrence of linear power of z as well as terms of the type $z \cos\left(\frac{j_n z}{\beta}\right)$ in the fourth order perturbations renders the present method invalid for the large values of z. However, we have checked carefully that, up to the values of z and for the values of ϵ that we have considered in the paper, these terms do not produce any difficulty except that the presence of sine and cosine terms in the solution imparts it an oscillatory character. This fact is exhibited by Table II also.

To sum up the findings of the present investigation, we can say that we are able to explain, at least qualitatively, the experimental results of Laven and Fejer.

Acknowledgement

One of us (P.L.B.) is grateful to the N.R.C. of Canada for a grant under which this investigation has been undertaken and to the authorities of the University of Waterloo for the kind hospitality and facilities provided to him during his stay at the Department of Applied Mathematics. We are highly obliged to Mr. D. K. Vijay of Civil Engineering and Dr. H. C. Wong of Applied Analysis and Computer Science Department of the University of Waterloo for undertaking heavy computational work on the digital computer of the University.

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Appendix A

$$\int_0^1 tJ_1(j_s t)J_1(j_n t)dt = 0$$

$$\int_0^1 t^2 J_0(j_n t)J_1(j_s t)dt = \frac{-j_s}{j_s^2 - j_n^2} J_0(j_n)J_0(j_s) \qquad \ldots \text{(A2)}$$

$$\int_0^1 t^3 J_1(j_n t)J_1(j_s t)dt = \frac{4j_s j_n}{(j_s^2 - j_n^2)^2} J_0(j_s)J_0(j_n) \qquad \ldots \text{(A3)}$$

$$\int_0^1 t^4 J_0(j_n t)J_1(j_n t)dt = -\frac{J_0^2(j_n)}{6j_n} \qquad \ldots \text{(A4)}$$

$$\int_0^1 t^3 J_0^2(j_n t)dt = \frac{1}{6} J_0^2(j_n) \qquad \ldots \text{(A5)}$$

$$\int_0^r tJ_1^2(j_n t)dt = \frac{r^2}{2}\left[J_0^2(j_n r) + J_1^2(j_n r) - \frac{2}{j_n r} J_1(j_n r)J_0(j_n r)\right] \qquad \ldots \text{(A6)}$$

$$\int_0^r t^3 J_1^2(j_n t)dt = \frac{r^4}{6}[J_1^2(j_n) + J_2^2(j_n r)] \qquad \ldots \text{(A7)}$$

$$\int_0^r t^2 J_0(j_n t)J_1(j_n t)dt = \frac{r^2 J_1^2(j_n r)}{2j_n} . \qquad \ldots \text{(A8)}$$

APPENDIX B

Let

$$I = \int_0^x tJ_0(\alpha t)J_0(\beta t)J_0(\gamma t)dt \quad \ldots \text{(B1)}$$

and

$$I_{011} = \int_0^x tJ_0(\alpha t)J_1(\beta t)J_1(\gamma t)dt \quad \ldots \text{(B2)}$$

then on integrating by parts the latter integral with the help of the Lommel integral

$$(\lambda^2-\mu^2)\int_0^x tJ_n(\lambda t)J_n(\mu t)dt = \mu x J'_n(\mu x)J_n(\lambda x)-\lambda x J_n(\mu x)J'_n(\lambda x) \quad \ldots \text{(B3)}$$

and simplifying, we get

$$2\beta\gamma\, I_{011} = \alpha x\, J_1(\alpha x)\, J_0(\beta x)\, J_0(\gamma x)-\beta x\, J_1(\beta x)\, J_0(\gamma x)\, J_0(\alpha x) - \gamma x\, J_1(\gamma x)\, J_0(\alpha x)\, J_0(\beta x)+(\beta^2+\gamma^2+\alpha^2)I \quad \ldots \text{(B4)}$$

with symmetrical expressions for I_{101} and I_{110}.

Let

$$I_{111} = \int_0^x J_1(\alpha t)J_1(\beta t)J_1(\gamma t)dt. \quad \ldots \text{(B5)}$$

Integrating by parts, we get

$$2I_{111} = -xJ_1(\alpha x)J_1(\beta x)J_1(\gamma x)+\alpha I_{011}+\beta I_{101}+\gamma I_{110}. \quad \ldots \text{(B6)}$$

Substituting the expressions for the integrals occurring on the right hand side of (B6) from (B4) and simplifying, we get

$$I_{111} = -\frac{x}{2}\, J_1(\alpha x)J_1(\beta x)J_1(\gamma x)+\frac{x}{4\alpha\beta\gamma}[\alpha(\alpha^2-\beta^2-\gamma^2)J_1(\alpha x)J_0(\beta x)J_0(\gamma x) +\beta(\beta^2-\gamma^2-\alpha^2)J_0(\alpha x)J_1(\beta x)J_0(\gamma x)+\gamma(\gamma^2-\alpha^2-\beta^2)J_0(\alpha x)J_0(\beta x). \times J_1(\gamma x)]+\frac{2(\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2)-(\alpha^4+\beta^4+\gamma^4)}{4\alpha\beta\gamma}\, I. \quad \ldots \text{(B7)}$$

Let now

$$\bar{I} = \int_x^1 tJ_0(\alpha t)J_0(\beta t)Y_0(\gamma t)dt \quad \ldots \text{(B8)}$$

and

$$\bar{I}_{110} = \int_x^1 tJ_1(\alpha t)J_1(\beta t)Y_0(\gamma t)dt \quad \ldots \text{(B9)}$$

then on integrating (B9) by parts with the help of (B3) we get

$$2\beta\gamma\, \bar{I}_{001} = \gamma x J_0(\alpha x)J_0(\beta x)Y_1(\gamma x)+\beta x J_0(\alpha x)J_1(\beta x)Y_0(\gamma x)-\alpha x J_1(\alpha x)J_0(\beta x). \times Y_0(\gamma x)-\gamma J_0(\alpha)J_0(\beta)Y_1(\gamma)+(\beta^2+\gamma^2-\alpha^2)\,\bar{I}. \quad \ldots \text{(B10)}$$

Similarly

$$2\alpha\beta\,\bar{I}_{110} = -\gamma x J_0(\alpha x)J_0(\beta x)Y_1(\gamma x)+\beta x J_0(\alpha x)J_1(\beta x)Y_0(\gamma x)+\alpha x J_1(\alpha x)J_0(\beta x).$$
$$\times Y_0(\gamma x)+\gamma J_0(\alpha)J_0(\beta)Y_1(\gamma)+(\alpha^2+\beta^2-\gamma^2)\,\bar{I}. \qquad \ldots \text{(B11)}$$

and

$$\bar{I}_{111} = \int_x^1 J_1(\alpha x)J_1(\beta x)Y_1(\gamma t)dt = \frac{x}{2}\,J_1(\alpha x)J_1(\beta x)Y_1(\gamma x)+\frac{x}{4\alpha\beta\gamma}[\gamma(\alpha^2+\beta^2-\gamma^2)$$
$$\times J_0(\alpha x)J_0(\beta x)Y_1(\gamma x)+\beta(\alpha^2+\gamma^2-\beta^2)J_0(\alpha x)J_1(\beta x)Y_0(\gamma x)$$
$$+\alpha(\gamma^2+\beta^2-\alpha^2)J_1(\alpha x)J_0(\beta x)Y_0(\gamma x)]+\frac{\gamma(\gamma^2-\alpha^2-\beta^2)}{4\alpha\beta\gamma}\,J_0(\alpha)J_0(\beta)$$
$$\times Y_1(\gamma)+\frac{I}{4\alpha\beta\gamma}\,[2(\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2)-(\alpha^4+\beta^4+\gamma^4)]. \qquad \ldots \text{(B12)}$$

In a separate paper*, one of us (P.L.B.) has evaluated analytically much more general integrals than recorded here and has numerically integrated I and $\bar{I}$ when α, β, γ are the zeros j_m, j_n, j_s of $J_1(z)$ and their combinations like $j_m \pm j_n$ for all values of m, n, s from 1 to 5.

*To be published in *Comment. math. Univ. S. Pauli Tokyo*, Vol. **XXII**, No. 1.