ON PLANE WAVE SOLUTIONS OF THE FIELD EQUATIONS OF GENERAL RELATIVITY

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Takeno has obtained the plane wave solutions of the field equations of general relativity defining them to be the non-flat solutions of the field equations with the additional condition that in some suitable coordinate system all the components of the metric tensor are functions of a single variable $Z(z-t)$, the phase function. In the present paper a generalisation of the plane wave solutions has been attempted by assuming the space-time metric with all its components as functions of a single variable $Z(z, t)$. This approach has the advantage that the general solutions so obtained reduce to different types of plane wave solutions under restricted conditions.

1. INTRODUCTION

Takeno (1961) has investigated the $(z-t)$-type plane wave solutions of the field equations of general relativity

$$G_{ij} = 8\pi E_{ij}, \qquad (i, j = 1, 2, 3, 4), \qquad \ldots \qquad \ldots \quad (1.1)$$

where G_{ij} is the Ricci tensor of the space-time and E_{ij} is the electromagnetic energy tensor defined by

$$E_{ij} = \frac{1}{4} g_{ij} F_{kl} F^{kl} - F_{ik} F_{jl} g^{kl}. \qquad \ldots \qquad \ldots \quad (1.2)$$

Here g_{ij} is the fundamental tensor of the space-time and F_{ij} is the anti-symmetric electromagnetic field tensor satisfying the generalized Maxwell's equations

$$\text{(i)} \quad F_{ij;k} + F_{jk;i} + F_{ki;j} = 0, \qquad \text{(ii)} \quad F^{ij}_{\ ;j} = 0 \qquad \ldots \qquad \ldots \quad (1.3)$$

and a semicolon denotes a covariant derivative.

Takeno's plane wave solutions are based on the following definition :

(A) A plane wave g_{ij} is a non-flat solution of (1.1).

(B) A plane wave g_{ij} has the property that in some suitable coordinate system all its components are functions of a single variable $Z(= x-t)$.

In formulating the above definition he assumed the existence of a 'Cartesian-like' coordinate system and considered the waves propagating in the positive direction of z-axis with phase velocity equal to the fundamental velocity.

In the present paper, generalized plane wave solutions have been discovered by considering the space-time metric

$$ds^2 = -Adx^2-2Ddxdy-Bdy^2-\phi^2(C-E)dz^2+2\phi Edzdt+(C+E)dt^2 \qquad (1.4)$$

and using the above definition of a plane wave. Here A, B, C, D, E are functions of the single variable $Z = Z(z, t)$ and $\phi = Z_{,3}/Z_{,4}$, $\left(,i = \frac{\partial}{\partial x^i}\right)$. The quadratic form $dl^2 = Adx^2+2Ddxdy+Bdy^2$ is positive definite, ϕ is real, C is positive and greater than $|E|$. The usual signature conditions are obviously satisfied, since $A > 0$, $B > 0$, $m = (AB-D^2) > 0$ and $C > |E|$.

In order that the metric (1.4) should conform to the above Takeno's definition of a plane wave, the phase function Z is so chosen that it satisfies the relation $\phi = Z_{,3}/Z_{,4} = f(Z)$, i.e. a function of Z. Thus, all the components of g_{ij} are now functions of a single variable $Z = Z(z, t)$.

Having suitably chosen the phase function Z, the components of the curvature tensor, Ricci tensor, the electromagnetic field tensor and the electromagnetic energy tensor have been calculated in the following sections.

2. The Curvature Tensor and Ricci Tensor

To calculate the components of the curvature tensor and the Ricci tensor in the space-time defined by the metric (1.4), first the components of the Christoffel symbols are written down, which are calculated from the line element (1.4) as follows :

The components of the fundamental tensor g_{ij} from (1.4) are :

$$\left.\begin{aligned} &g_{11} = -A, && g_{33} = -\phi^2(C-E) \\ &g_{12} = -D, && g_{34} = \phi E \\ &g_{22} = -B, && g_{44} = (C+E) \\ \text{other}\quad &g_{ij} = 0. \end{aligned}\right\} \quad \ldots \quad \ldots \quad (2.1)$$

The components of the corresponding contravariant tensor g^{ij} are :

$$\left.\begin{aligned} &g^{11} = -B/m, && g^{34} = E/\phi C^2 \\ &g^{12} = D/m, && g^{33} = -(C+E)/\phi^2 C^2 \\ &g^{22} = -A/m, && g^{44} = (C-E)/C^2 \\ \text{other}\quad &g^{ij} = 0. \end{aligned}\right\} \quad \ldots \quad (2.2)$$

Also,

$$g = (\det.g_{ij}) = -(AB-D^2)\phi^2C^2 = -mC^2\phi^2. \quad \ldots \quad (2.3)$$

The components of the Christoffel symbol of the first kind can easily be calculated from (2.1) and (2.2) which are omitted for brevity's sake. The components of the Christoffel symbol of the second kind are :

$$\left.\begin{aligned}
\{11,k\} &= \left[\ 0\ ,\quad 0\ ,\quad -\frac{\bar{A}v}{2\phi C}\ ,\quad \frac{\bar{A}v}{2C}\ \right]\\
\{12,k\} &= \left[\ 0\ ,\quad 0\ ,\quad -\frac{\bar{D}v}{2\phi C}\ ,\quad \frac{\bar{D}v}{2C}\ \right]\\
\{13,k\} &= [\ Pu\ ,\quad Su\ ,\quad 0\ ,\quad 0\ ]\\
\{14,k\} &= [\ Pv\ ,\quad Sv\ ,\quad 0\ ,\quad 0\ ]\\
\{22,k\} &= \left[\ 0\ ,\quad 0\ ,\quad -\frac{\bar{B}v}{2\phi C}\ ,\quad \frac{\bar{B}v}{2C}\ \right]\\
\{23,k\} &= [\ Qu\ ,\quad Tu\ ,\quad 0\ ,\quad 0\ ]\\
\{24,k\} &= [\ Qv\ ,\quad Tv\ ,\quad 0\ ,\quad 0\ ]\\
\{33,k\} &= \left[\ 0\ ,\ 0\ ,\ \left(\frac{\bar{\phi}}{\phi}+\frac{E\bar{\phi}}{C\phi}+\frac{\bar{C}}{2C}-R\right)u,\ \left(\bar{\phi}-\frac{2E\bar{\phi}}{C}+\frac{\bar{C}\bar{\phi}}{2C}+\phi R\right)u\ \right]\\
\{34,k\} &= \left[\ 0\ ,\ 0\ ,\ \left(\frac{\bar{\phi}}{\phi}+\frac{\bar{C}}{2C}-R\right)v,\ \left(-\frac{E\bar{\phi}}{C\phi}+\frac{\bar{C}}{2C}+R\right)u\ \right]\\
\{44,k\} &= \left[\ 0\ ,\ 0\ ,\ \left(\frac{\bar{C}}{2C\phi}-\frac{E\bar{\phi}}{C\phi^2}-\frac{R}{\phi}\right)v,\ \left(\frac{\bar{C}}{2C}+R\right)v\ \right]
\end{aligned}\right\} \quad \ldots \quad (2.4)$$

where a bar over any symbol denotes a derivative with respect to $Z(z,t)$, $u = Z_{,3}$, $v = Z_{,4}$ and

$$\left.\begin{aligned}
P &= (\bar{A}B-\bar{D}D)/2m, \qquad Q = (B\bar{D}-\bar{B}D)/2m\\
S &= (A\bar{D}-\bar{A}D)/2m, \qquad T = (A\bar{B}-D\bar{D})/2m\\
R &= (\bar{E}/2C+E^2\bar{\phi}/C^2\phi-E\bar{C}/C^2).
\end{aligned}\right\} \quad \ldots \quad \ldots \quad (2.5)$$

Using the components of the Christoffel symbols, the components of the curvature tensor G_{ijkl} are easily calculated and its non-vanishing components are:

$$\left.\begin{aligned}
&G_{1313}/u^2 = G_{1314}/uv = G_{1414}/v^2 = L\\
&G_{2323}/u^2 = G_{2324}/uv = G_{2424}/v^2 = M\\
&G_{1323}/u^2 = G_{1324}/uv = G_{1423}/uv = G_{1424}/v = N\\
&G_{3434} = C\phi\phi_{,44}
\end{aligned}\right\} \quad \ldots \quad \ldots \quad (2.6)$$

where

$$\left.\begin{aligned} 2L &= \left[\bar{\bar{A}} - \frac{\bar{A}\bar{C}}{C} - \frac{B\bar{A}^2 + A\bar{D}^2 - 2D\bar{A}\bar{D}}{2m} + \frac{\bar{A}E\bar{\phi}}{C\phi} + \bar{A}\left(\frac{v_{,4}}{v^2}\right)\right] \\ 2M &= \left[\bar{\bar{B}} - \frac{\bar{B}\bar{C}}{C} - \frac{A\bar{B}^2 + B\bar{D}^2 - 2D\bar{B}\bar{D}}{2m} + \frac{\bar{B}E\bar{\phi}}{C\phi} + \bar{B}\left(\frac{v_{,4}}{v^2}\right)\right] \\ 2N &= \left[\bar{\bar{D}} - \frac{\bar{D}\bar{C}}{C} - \frac{A\bar{B}\bar{D} + \bar{A}B\bar{D} - \bar{A}\bar{B}D - D\bar{D}^2}{2m} + \frac{\bar{D}E\bar{\phi}}{C\phi} + \bar{D}\left(\frac{v_{,4}}{v^2}\right)\right] \end{aligned}\right\} \quad \ldots \quad (2.7)$$

The non-vanishing components of the Ricci tensor obtained from (2.6) on contraction, are as follows :

$$\left.\begin{aligned} G_{33} &= u^2\left(W - \frac{C-E}{Cuv}\,\phi_{,44}\right) \\ G_{34} &= uv\left(W + \frac{E}{Cuv}\,\phi_{,44}\right) \\ G_{44} &= v^2\left(W + \frac{C+E}{Cuv}\,\phi_{,44}\right) \end{aligned}\right\} \quad \ldots \quad \ldots \quad (2.8)$$

where

$$W = \frac{\bar{\bar{m}}}{2m} - \frac{\bar{m}^2}{4m^2} - \frac{\bar{m}\bar{C}}{2mC} - \frac{\bar{A}\bar{B} - \bar{D}^2}{2m} + \frac{E\bar{m}\bar{\phi}}{2mC\phi} + \frac{\bar{m}}{2m}\left(\frac{v_{,4}}{v^2}\right).$$

3. The Electromagnetic Field

Considering the components of the electromagnetic potential $k_i(-F, -G, -H, \chi)$ as functions of $Z = Z(z, t)$, where (F, G, H) is the vector potential and χ the scalar potential of the electromagnetic field F_{ij}, which is described by the components of the electric force and the components of the magnetic force. This in tensor notation can be put as

$$F_{ij} = \frac{\partial k_i}{\partial x^j} - \frac{\partial k_j}{\partial x^i}\,, \quad (F_{ij} = -F_{ji} \text{ and } x^1, x^2, x^3, x^4 = x, y, z, t) \quad \ldots \quad (3.1)$$

The transverse character of the electromagnetic waves propagating along the positive direction of z-axis requires the vanishing of the components of the electric and the magnetic force in this direction, which gives

$$F_{34} = F_{12} = 0. \quad \ldots \quad \ldots \quad (3.2)$$

This imposes a condition on the components of the electromagnetic potential, namely

$$v\bar{H}+u\bar{X}=0. \qquad \ldots \quad \ldots \quad (3.3)$$

Thus, the non-vanishing components of F_{ij} are as follows :

$$\left.\begin{aligned} F_{31}/u &= -F_{14}/v = \bar{F} = \rho, \\ F_{32}/u &= -F_{24}/v = \bar{G} = \sigma, \end{aligned}\right\} \qquad \ldots \quad \ldots \quad (3.4)$$

where ρ and σ are arbitrary functions of $Z = Z(z, t)$.

The non-vanishing components of the contravariant tensor F^{ij} obtained from (2.2) and (3.4) are as follows :

$$\left.\begin{aligned} F^{31}/v &= F^{14}/u = (B\rho-D\sigma)/mC\phi \\ F^{23}/v &= F^{42}/u = (D\rho-A\sigma)/mC\phi. \end{aligned}\right\} \qquad \ldots \quad \ldots \quad (3.5)$$

The components of the dual tensor F^*_{ij} of the electromagnetic field tensor F_{ij} are given by $F^*_{ij} = \frac{1}{2}E_{ijkl}F^{kl}$, where E_{ijkl} is anti-symmetric with respect to each pair of indices and $E_{1234} = \sqrt{-g} = \sqrt{m}C\phi$. The components of F^*_{ij} are easily calculated. They are

$$\left.\begin{aligned} F^*_{13}/u &= F^*_{14}/v = (D\rho-A\sigma)/\sqrt{m} \\ F^*_{23}/u &= F^*_{24}/v = (B\rho-D\sigma)/\sqrt{m}. \end{aligned}\right\} \qquad \ldots \quad \ldots \quad (3.6)$$

From (3.4), (3.5) and (3.6), the following relation is obtained :

$$F_{ij}F^{ij} = F^*_{ij}F^{ij} = 0 \qquad \ldots \quad \ldots \quad (3.7)$$

which shows that the F_{ij} is null in the sense of Synge (1956).

Next, on substituting the values from (2.1), (2.2), (3.4) and (3.5) into (1.2), the following non-vanishing components of electromagnetic energy tensor E_{ij} are obtained :

$$E_{33}/u^2 = E_{34}/uv = E_{44}/v^2 = (A\sigma^2+B\rho^2-2D\rho\sigma)/m. \qquad \ldots \quad (3.8)$$

4. Exact Plane Wave Solutions of the Field Equations

Substituting the values of G_{ij} and E_{ij} from (2.8) and (3.8) into the field equation (1.1), the following three equations are obtained :

$$\left(W-\frac{C-E}{Cuv}\phi_{,44}\right) = -\frac{8\pi}{m}(A\sigma^2+B\rho^2-2D\rho\sigma) \qquad \ldots \quad \ldots \quad (4.1a)$$

$$\left(W+\frac{E}{Cuv}\phi_{,44}\right) = -\frac{8\pi}{m}(A\sigma^2+B\rho^2-2D\rho\sigma) \qquad \ldots \quad \ldots \quad (4.1b)$$

$$\left(W+\frac{C+E}{Cuv}\phi_{,44}\right) = -\frac{8\pi}{m}(A\sigma^2+B\rho^2-2D\rho\sigma) \qquad \ldots \quad \ldots \quad (4.1c)$$

Evidently, they are satisfied under the conditions

$$W = -\frac{8\pi}{m}(A\sigma^2+B\rho^2-2D\rho\sigma)$$

or

$$\bar{\bar{m}}-\frac{\bar{m}^2}{2m}-\frac{\bar{m}\bar{C}}{C}-(\bar{A}\bar{B}-\bar{D}^2)+\frac{E\bar{m}\bar{\phi}}{C\phi}+\bar{m}\left(\frac{v_{,4}}{v^2}\right) = -16\pi(A\sigma^2+B\rho^2-2D\rho\sigma) \quad \ldots (4.2a)$$

and

$$\phi_{,44} = 0. \quad \ldots \quad \ldots (4.2b)$$

The field equations (1.3) are identically satisfied by the components of F_{ij} and F^{ij} given by (3.4) and (3.5) respectively.

Therefore, the plane wave solutions of the field equations (1.1), (1.3) in the space-time whose metric is (1.4), are composed of the g_{ij} given by (2.1) and the F_{ij} given by (3.4), under the conditions (4.2).

The plane wave solutions obtained above contain in them the solutions of $(z-t)$-type, (t/z)-type and (z/t)-type (Johri 1967) plane waves, which are readily obtained, by taking $\phi = -1$, $-Z$ and $-1/Z$ respectively.

References

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