QUANTITATIVE STABILITY ANALYSIS OF INTEGRO-DIFFERENTIAL EQUATIONS IN BANACH SPACES

by B. G. PACHPATTE, *Department of Mathematics, Deogiri College, Aurangabad (Maharashtra)*

(Communicated by B. R. Seth, F.N.A.)

(*Received 14 April 1972*)

The purpose of this paper is to develop the quantitative stability analysis of integro-differential equations in Banach spaces, within the framework of the 'Lyapunov technique' which basically depends on the comparison principle. After stating various definitions of quantitative stability, theorems that yield sufficient conditions for various forms of stability are stated and proved. These theorems involve the existence of Lyapunov-like functions which in general do not possess the usual definiteness requirements on such functions or their derivative.

1. INTRODUCTION

It is well known that the theory of one parameter semi-groups of linear and bounded operators developed by Kato (1956) has been well used to investigate the existence and uniqueness theorems for the solutions of linear differential equations in Banach spaces. Krasnoselskii *et al.* (1956) have also considered such equations including nonlinear differential equations in Banach spaces. Mlak (1959) obtained some results concerning the limitations of solutions of nonlinear differential equations in Banach spaces. Lakshmikantham and Leela (1969) have used the second method of Lyapunov to study the stability and boundedness of solutions of differential equations in Banach spaces. In this paper, a quantitative treatment of integro-differential equations is considered. Incorporating an idea used by LaSalle and Lefschetz (1961) in the discussion of 'practical stability', recently, several forms of stability of a more quantitative nature than most classical stability definitions have been presented by various investigators. Following Michel (1970), we extend the definitions of various types of stability in terms of time-varying sets which are pre-specified in a given problem. The properties of these sets not only yield stability information but they also yield estimates of trajectory bounds and trajectory behaviour of integro-differential equations in Banach spaces. As is usually the case in stability theory, we develop

sufficient conditions for quantitative stability in terms of the existence of Lyapunov-like functions. The theory developed thus far parallels, to a certain extent, the classical Lyapunov theory of stability, but differs from it in a number of significant respects which are evident from the definitions and theorems.

2. Definitions and Preliminaries

Let B be a real Banach space. We are interested in the operators which are defined on subsets of B and take the values belonging to B. The operator A is linear if it is additive and homogeneous. Denote its domain of definition by $D[A]$. The operator A is said to be closed if the set of couples (x, Ax), $x \in D[A]$ is closed in the topological product $B \times B$. For a real λ, suppose that $[I-\lambda A]^{-1}$ exists, where I is the identity operator and $D[I-\lambda A]^{-1}$ is dense in B. Let $R(\lambda, A) = [I-\lambda A]^{-1}$ and since A is closed we have

$$D[R(\lambda, A)] = B$$

$$[I-\lambda A]R(\lambda, A)x = x \qquad (x \in B)$$

and

$$R(\lambda, A)[I-\lambda A]x = x, \qquad x \in D[A]$$

where $R(\lambda, A)$ is called the resolvent of A.

Let J denote the interval $t_0 \leqslant t < \infty$, $t_0 \geqslant 0$. Let $\|\cdot\|$ denotes any suitable norm in B. We denote by $C[M, F]$, the class of functions defined and continuous on M taking values in F, where M and F are any convenient spaces. Let $C(J)$ denote the space of continuous functions $u \in C[J, B]$ and E be a continuous operator such that E maps $C(J)$ into $C(J)$. For any two continuous functions $u, v \in C[J, B]$ the operator E is assumed to satisfy the following property :

$$u(t) \leqslant v(t), \; t_0 \leqslant t < t_1, \; t_1 \in (t_0, \infty)$$

implies

$$Eu \leqslant Ev \quad \text{for } t = t_1.$$

We consider the integro-differential system

$$x' = A(t)x + f(t, x, Ex) \qquad \ldots \quad (2.1)$$

where $f \in C[J \times B \times C(J), B]$, and the operator $A(t)$ satisfy the assumption (*) given below.

(*) Suppose that for each $t \in J$, $\{A(t)\}$ is a one parameter family of closed linear operators. Assume that for each $t \in J$, the resolvent set of $A(t)$ includes all positive real numbers, that the domain of $R(\lambda, A(t))$ is dense in B, and that the domain $D[A(t)]$ of $A(t)$ is independent of t.

Let $x(t) \in D[A(t)]$ be a strongly continuous function defined for $t \in J$ and having a strong derivative $x'(t)$ in B. We shall call $x(t)$ a solution of the integro-differential system (2.1) if it satisfies eqn. (2.1) for $t \in (t_0, \infty)$, with the initial value $x(t_0) = x_0$, $x_0 \in D[A(t_0)]$. We denote the solution of (2.1) through $(t_0, x_0) \in J \times B$ by $x(t, t_0, x_0)$.

We note that, if $f(t, x, y)$ in (2.1) is of the form

$$f(t, x, y) = h(t, x) + y$$

and if the operator E is defined by

$$Ex = \int_{t_0}^{t} K(t, s, x(s))ds$$

then the system (2.1) reduces to

$$x' = A(t)x + h(t, x) + \int_{t_0}^{t} K(t, s, x(s))ds$$

which contains, as a special case the integro-differential equation considered by Marti (1967).

On the other hand, if E is defined by $Ex = x$, then the system (2.1) reduces to the differential system considered by various investigators (Krasnoselskii *et al.* 1956, Lakshmikantham and Leela 1969, Mlak 1959).

Associated with (2.1) are various 'perturbed' integro-differential systems

$$x' = A(t)x + f(t, x, Ex) + g(t, x, Ex) \qquad \ldots \quad (2.2)$$

where the perturbing functions $g \in C[J \times B \times C(J), B]$. Solution of (2.2) through $(t_0, x_g(t_0)) \in J \times B$ will be denoted by $x_g(t, t_0, x_g(t_0))$.

Let $S(t) \subset B$ for all $t \in J$. Assume that $S(t)$ is a connected open region. Let $\overline{S(t)}$ denote the closure of $S(t)$ and let $\partial S(t)$ denote the boundary of $S(t)$. Assume that $S(t)$ is bounded for all $t \in J$, and $\lim_{t \to t_a} S(t)$ exists for all $t_a \in J$, and that $\lim_{t \to t_a} S(t) = S(t_a)$. Henceforth, whenever the symbol $S(t)$ (with appropriate subscripts) is used to denote a set, it is assumed that this set possesses the properties described above and G is some collection of continuous functions of the form $g \in C[J \times B \times C(J), B]$. Let t_i be any point (initial time) in J and let $x_i = x(t_i, t_i, x_i)$. Our interest lies in the following stability definitions.

Definition 1—The system (2.1) is 'stable' with respect to $\{S_0(t), S(t), t_0\}$, if $x_0 \in S_0(t_0)$ implies $x(t, t_0, x_0) \in S(t)$ for all $t \in J$.

Definition 2—The system (2.1) is 'uniformly stable' with respect to $\{S_0(t), S(t)\}$, if for all $t_i \in J$, $x_i \in S_0(t_i)$ implies $x(t, t_i, x_i) \in S(t)$ for each $t \in [t_i, \infty)$.

Definition 3—The system (2.1) is 'totally stable' with respect to $\{S_0(t), S(t), G, t_0\}$, if whenever $g \in G$, the condition $x_g(t_0) \in S_0(t_0)$ implies $x_g(t, t_0, x_g(t_0)) \in S(t)$ for all $t \in J$.

Definition 4—The system (2.1) is 'uniformly totally stable' with respect to $\{S_0(t), S(t), G\}$, if whenever $g \in G$, then for each $t_i \in J$, the condition $x_g(t_i) \in S_0(t_i)$ implies $x_g(t, t_i, x_g(t_i)) \in S(t)$ for all $t \in [t_i, \infty)$.

The stability theorems presented in our subsequent discussion make use of Lyapunov-like functions of the form $V = V(t, x) : J \times B \to R^1$. The function V is said to belong to the class C^1, if it is continuous in both t and x and satisfies for each $t \in J$, a local Lipschitz condition in x. Also, the function $W = W(t, u, v) : J \times R^1 \times R^1 \to R^1$ is said to belong to the class Ω, if it is continuous in t, u and v, monotone increasing in v for any fixed t and u and it is smooth enough to ensure the existence of a maximal solution of the scalar integro-differential equation

$$r' = W(t, r, Er),\ r(t_0) = r_0, t \in J. \qquad \ldots \quad (2.3)$$

We define

$$V^*(t, x, y) = \lim_{h \to 0^+} \sup \frac{1}{h} [V(t+h, R(h, A(t))x + hf(t, x, y)) - V(t, x)]. \qquad \ldots \quad (2.4)$$

We require the following lemma which can be found in a paper by Sugiyama (1965). We shall state it in a suitable form whose proof needs very little modification of the proof given in the theory of ordinary differential equations (see Lakshmikantham and Leela 1969).

Lemma 1—Let the function W be as explained above. Suppose that $r(t, t_0, r_0)$ is the maximal solution of the integro-differential equation (2.3) existing to the right of t_0. If there exists a function $m(t) \in C[J, R^1]$ satisfying the inequalities

$$m(t_0) \leqslant r(t_0)$$

and

$$m'(t) < W(t, m(t), Em(t))$$

then

$$m(t) < r(t, t_0, r_0) \text{ for all } t \in J.$$

3. Main Results

In this section theorems are stated and proved which yield sufficient conditions for various forms of stability in the sense of the definitions of section 2. These conditions involve the existence of 'Lyapunov-like'

functions whose properties differ significantly from those of classical Lyapunov functions.

Theorem 1—System (2.1) is stable with respect to $\{S_0(t), S(t), t_0\}$, where $S_0(t) \subset S(t)$ and $\partial S(t) \bigcap \partial S_0(t) = \phi$ for all $t \in J$, if there exists a function $V(t, x)$ which is in C^1 for all $x \in S(t)$ and for all $t \in J$, and a function $W \in \Omega$ which satisfies the assumptions of Lemma 1, such that

$$\text{(i)} \qquad \lim_{h \to 0^+} R(h, A(t))x = x, \quad x \in S(t)$$

$$\text{(ii)} \qquad V^*(t, x, Ex) < W(t, V(t, x), (EV)(t, x)), x \in S(t)$$

$$\text{(iii)} \qquad r(t, t_0, \sup_{x \in S_0(t_0)} V(t_0, x)) \leqslant \inf_{x \in \partial S(t)} V(t, x),$$

where $r(t, t_0, r_0)$ is the maximal solution of (2.3).

Proof : The proof is by contradiction. Let $x(t) = x(t, t_0, x_0)$ be a solution of (2.1) with $x_0 \in S_0(t_0)$. Assume that there exists a $t_1 \in (t_0, \infty)$, the first time such that $x(t_1) \in \partial S(t_1)$. Now define $m(t) = V(t, x(t))$. Further for small $h > 0$,

$$\begin{aligned} m(t+h)-m(t) \leqslant c[\, &\|x(t+h)-R(h, A(t))x(t)-hf(t, x(t), Ex(t))\| \,] \\ &+ V(t+h, R(h, A(t))x(t)+hf(t, x(t), Ex(t)))-V(t, x(t)) \qquad \ldots \quad (3.1) \end{aligned}$$

where $c > 0$, is a Lipschitz constant. Since for every $x \in D[A(t)]$, $R(h, A(t))\ (I-h\,A(t))x = x$, it follows that

$$\begin{aligned} R(h, A(t))x+hf(t, x, Ex) = x &+h[A(t)x+f(t, x, Ex)] \\ &+h[R(h, A(t))\ A(t)x-A(t)x]. \end{aligned}$$

This together with (3.1) implies that

$$\begin{aligned} m(t+h)-m(t) \leqslant c[&\|x(t+h)-x(t)-h\{A(t)x(t)+f(t, x(t), Ex(t))\}\|] \\ &+hc[\|R(h, A(t))\ A(t)x(t)-A(t)x(t)\|] \\ &+ V(t+h, R(h, A(t))x(t)+hf(t, x(t), Ex(t)))-V(t, x(t)). \end{aligned}$$

Using (2.1), (2.4), (i), (ii), we obtain the inequality

$$\lim_{h \to 0^+} \sup \frac{1}{h} [m(t+h)-m(t))] < W(t, m(t), Em(t)).$$

Now

$$m(t_0) = V(t_0, x(t_0)) \leqslant \sup_{x \in S_0(t_0)} V(t_0, x)$$

and thus it follows from Lemma 1 that

$$V(t, x(t)) < r(t, t_0, \sup_{x \in S_0(t_0)} V(t_0, x))$$

for all $t \in (t_0, t_1]$. In view of hypothesis (iii) one can now write

$$V(t_1, x(t_1)) < \inf_{x \in S_0(t_1)} V(t_1, x).$$

But this last inequality implies that $x(t_1) \notin \partial S(t_1)$, a contradiction to the original assumption. Therefore $x(t) \in S(t)$ for all $t \in J$.

Remark: The V function utilized in Theorem 1 is in general not a Lyapunov function and as such need not possess any particular definiteness requirements.

The same is true for the following theorems which deal with uniform stability, total stability and uniform total stability of (2.1).

Theorem 2—System (2.1) is uniformly stable with respect to $\{S_0(t), S(t)\}$, where $S_0(t) \subset S(t)$ and $\partial S(t) \cap \partial S_0(t) = \phi$ for all $t \in J$, if there exists a function $V(t, x)$ which is in C^1 for all $x \in [S(t) - \overline{S_0(t)}]$ and for all $t \in J$, and a function $W \in \Omega$ which satisfies the assumptions of Lemma 1, such that

(i) $\lim_{h \to 0^+} R(h, A(t))x = x, \ x \in S(t),$

(ii) $V^*(t, x, Ex) < W(t, V(t, x), (EV)(t, x)), \ x \in [S(t) - \overline{S(t)}],$

(iii) $r(t_2, t_1, \sup_{x \in \partial S_0(t_1)} V(t_1, x)) \leqslant \inf_{x \in \partial S(t_2)} V(t_2, x), \ t_1, t_2 \in J, \ t_2 > t_1,$

where $r(t, t_0, r_0)$ is the maximal solution of (2.3).

The proof of this theorem is analogous to the proof of Theorem 1. We omit the details.

Theorem 3—System (2.1) is totally stable with respect to $\{S_0(t), S(t), G, t_0\}$, where $S_0(t) \subset S(t)$, and $\partial S_0(t) \cap \partial S(t) = \phi$ for all $t \in J$, if there exists a function $V(t, x)$ which is in C^1 for all $x \in S(t)$ and for all $t \in J$, and a function $W \in \Omega$ which satisfies the assumptions of lemma 1, such that

(i) $\lim_{h \to 0^+} R(h, A(t))x = x, \ x \in S(t),$

(ii) $\|g(t, x, Ex)\| \leqslant nV(t, x), x \in S(t), (n > 0),$

(iii) $V^*(t, x, Ex) + \alpha V(t, x) < W(t, V(t, x), (EV)(t, x)), x \in S(t)$

(iv) $r(t, t_0, \sup_{x \in S_0(t_0)} V(t_0, x)) \leqslant \inf_{x \in \partial S_0(t)} V(t, x)$

where $\alpha = nc$ and $r(t, t_0, r_0)$ is the maximal solution of (2.3).

Proof : For small $h > 0$, we have

$$V(t+h, R(h, A(t))x+h\{f(t, x, Ex)+g(t, x, Ex)\})-V(t, x)$$
$$\leqslant ch[\|g(t, x, Ex)\|]+V(t+h, R(h, A(t))x+f(t, x, Ex))-V(t, x).$$

This together with the assumptions (ii), (iii) yields, the inequality

$$V^{**}(t, x, Ex) < W(t, V(t, x), (EV)(t, x))$$

where

$$V^{**}(t, x, Ex) = \lim_{h \to 0^+} \sup \frac{1}{h}[V(t+h, R(h, A(t))x+h\{f(t, x, Ex)+g(t, x, Ex)\})$$
$$-V(t, x)].$$

Now, the proof of Theorem 1 is directly applicable and the stated result follows.

Theorem 4—The system (2.1) is uniformly totally stable with respect to $\{S_0(t), S(t), G\}$, where $S_0(t) \subset S(t)$ and $\partial S_0(t) \bigcap \partial S(t) = \phi$ for all $t \in J$, if there exists a function $V(t, x)$ which is in C^1 for all $x \in [S(t)-S_0(t)]$ and for all $t \in J$, and a function $W \in \Omega$ which satisfies the assumptions of Lemma 1 such that

(i) $\lim_{h \to 0^+} R(h, A(t))x = x, x \in S(t)$

(ii) $\|g(t, x, Ex)\| \leqslant nV(t, x), x \in S(t), (n > 0)$

(iii) $V^*(t, x, Ex)+\alpha V(t, x) < W(t, V(t, x), (EV)(t, x)), x \in [S(t)-\overline{S(t)}]$

(iv) $r(t_2, t_1, \sup_{x \in \partial S_0(t_1)} V(t_1, x)) \leqslant \inf_{x \in \partial S(t_2)} V(t_2, x), t_1, t_2 \in J, t_2 > t_1$

where $\alpha = cn$ and $r(t, t_0, r_0)$ is the maximal solution of (2.3).

The proof of this theorem can be formulated by following proofs of the above theorems, and we leave the details to the reader.

Remark : Two important special cases which are covered by above theorems are the abstract integro-differential system

$$x' = A(t)x+f(t)+\int_{t_0}^{t} a(t-s)x(s)ds$$

considered by Marti (1967) as well as the differential systems considered by various investigators (Krasnoselskii *et al.* 1956, Lakshmikantham and Leela 1969, Mlak 1959).

4. Applications

We give below an example to illustrate some of the silent features of quantitative stability analysis of integro-differential equations in Banach spaces.

Example—Consider the integro-differential system

$$x' = A(t)x+x+ \int_{t_0}^{t} k(t, s)\, x(s)ds \qquad \ldots \quad (4.1)$$

where $x \in B$, $k \in C[J \times J, R^1]$ and $A(t)$ is the operator which satisfies the assumption (*) given in section 2, and is such that $\| A(t)x \| < \| x \|$, and $\| k(t, s) \| \leqslant b(t)$, where $b(t) \in C[J, R^1]$.

We choose

$$V(t, x) = V(x) = \| x \|.$$

Now in view of the definition of V^* given in (2.4) together with the assumption (i) of Theorem 1, we observe that

$$V^*(t, x, Ex) < 2 \| x(t) \| + b(t) \int_{t_0}^{t} \| x(s) \|\, ds.$$

Thus the function W in (2.3) takes the form

$$W(t, r, Er) = 2r(t)+b(t) \int_{t_0}^{t} r(s)ds,$$

hence eqn. (2.3) reduces to

$$r'(t) = 2r(t)+b(t) \int_{t_0}^{t} r(s)ds \qquad \ldots \quad (4.2)$$

Now choose $S(t)$, $S_0(t)$ as

$$S(t) = \{x \in B : \| x \| < ke^{-(t-t_0)}-e^{-t}+e^{-t_0}\}$$

$$S_0(t) = \{x \in B : \| x \| < k_0e^{-(t-t_0)}-e^{-t}+e^{-t_0}\}$$

where $k_0 < k$, $k_0, k > 0$.

It can easily be shown that the maximal solution

$$r(t, t_0, r_0) = r_0e^{-(t-t_0)}-e^{-t}+e^{-t_0}$$

of (4.2) with the initial condition $r(t_0) = r_0$,
when

$$b(t) = \left[\frac{-3r_0e^{-(t-t_0)}+3e^{-t}-2e^{-t_0}}{r_0(1-e^{-(t-t_0)})+e^{-t}+e^{-t_0}(t-t_0-1)}\right]$$

satisfies the condition (iii) of Theorem 1. Thus the system (4.1) is stable with respect to $\{S(t), S_0(t), t_0\}$ for all $k > k_0 > 0$.

Remark : Theorems 1–4 yield sufficient conditions for stability, uniform stability, total stability and uniform total stability. No converse theorems have been developed yet for this type of stability, and thus the determination of the necessity of the stated hypotheses is an open problem.

REFERENCES

Kato, T. (1956). On linear differential equations in Banach spaces. *Communs pure appl. Math.*, **9**, 479–86.

Krasnoselskii, M. A., Krein, S. G., and Soboleveskii, P. E. (1956). On differential equations with unbounded operators in Banach spaces. *Dokl. Akad. Nauk. SSSR*, **111**, 19–22.

Lakshmikantham, V., and Leela, S. (1969). Differential and Integral Inequalities, Vol. I, II. Academic Press, New York.

LaSalle, J. P., and Lefschetz, S. (1961). Stability by Liapunov's Direct Method with Applications. Academic Press, New York.

Marti, J. T. (1967). On integro-differential equations in Banach spaces. *Pacific J. Math.*, **20**, 99–108.

Michel, A. N. (1970). Quantitative analysis of simple and interconnected systems : Stability, Boundedness, and Trajectory behaviour. *IEEE Trans. on Circuit Theory*, **17**, 292–301.

Mlak, M. (1959). Limitations and dependence on parameter of solutions of non-stationary differential operator equations. *Annls pol. math.*, 6, 305–22.

Sugiyama, S. (1965). On a certain functional-differential inequality. *Kodai math. Semin. Rep.*, **17**, 273–80.