ABSOLUTE RIESZ SUMMABILITY OF FOURIER SERIES AND ITS APPLICATION

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In this paper the author has obtained a new criterion for the absolute Riesz summability of Fourier series. This result has been used to obtain the convergence of Fourier series at a point $t = x$.

1. NOTATIONS AND INEQUALITIES

Let f be 2π-periodic and Lebesgue-integrable over $(-\pi, \pi)$. Let, without loss of generality, its Fourier series, at a point $t = x$, be

$$\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n=1}^{\infty} A_n(x).$$

For $\delta > 0$ and $\Delta > 1$, we write throughout

$$\phi(t) = \tfrac{1}{2}(f(x+t)+f(x-t)) \qquad \ldots \quad (1.1)$$

$$e(w) = \exp((\log w)^{\Delta}) \qquad \ldots \quad (1.2)$$

$$K(w, t) = \sum_{n \leq \omega} (e(w)-e(n))^{\delta}\, e(n) \cos nt \qquad \ldots \quad (1.3)$$

$$E(w, t) = \sum_{n \leq \frac{1}{2}\omega} (e(w)-e(n))^{\delta}\, e(n) \frac{\sin nt}{n} \qquad \ldots \quad (1.4)$$

$$\phi_1(t) = \frac{1}{t} \int_0^t \phi(u)du. \qquad \ldots \quad (1.5)$$

We use the following inequalities, which hold uniformly in $0 < t < \pi$, as $w \to \infty$.

$$E(w, t) = \begin{cases} \text{(i)} & O\{twe^{1+\delta}(w)(\log w)^{1-\Delta}\} \\ \text{(ii)} & O\{e^{1+\delta}(w)(\log w)^{1-\Delta}\} \\ \text{(iii)} & O\{t^{-1}w^{-1}e^{1+\delta}(w)\} \end{cases} \qquad \ldots \quad (1.6)$$

$$K(w, t) = O\{t^{-1-\delta}w^{-\delta}e^{1+\delta}(w)(\log w)^{\delta(\Delta-1)}\} + O\{t^{-2}w^{-1}e^{1+\delta}(w)(\log w)^{\Delta-1}\}. \qquad \ldots \quad (1.7)$$

For the proofs of (i), (ii) and (iii) of (1.6), comparison may be made with (3.1.2), (3.1.3) and (3.1.4) of Chandra (1970*a*) respectively. And for the proof of (1.7), comparison may be made with (3.2) of Chandra (1970*b*) and Chandra (1971, footnote to the p. 548).

2. Introduction

The following theorems are due to Mohanty (1951).

Theorem A—If $\phi(t) \log\log \frac{k}{t} \in BV(0,\pi)$, where $k \geqslant \pi e^2$, then $\sum_{n=1}^{\infty} A_n(x) \in |R, \exp\{(\log w)^{\Delta}\}, 1|$*($\Delta > 0$ however large).

Theorem B—If $\phi_1(t) \log \frac{k}{t} \in BV(0,\pi)$, where $k \geqslant \pi e^2$, then

$$\sum_{n=1}^{\infty} A_n(x) \in |R, \exp\{(\log w)^2, 2|.$$

The following theorem is a particular case of a more general theorem of Sinha (1958).

Theorem C—If $\phi_1(t) \left(\log \frac{k}{t}\right)^{\beta} \in BV(0,\pi)$, where $\beta \geqslant 0$ and $k \geqslant \pi e^{1+\beta}$, then $\sum_{n=1}^{\infty} A_n(x) \in |R, \exp\{(\log w)^{1+\beta}\}, 1+\delta|$ $(\delta > 0)$.

Theorem C generalizes Theorem B and a result of Bosanqauet (1936).

The object of this paper is two-fold : firstly to obtain a new criterion for the absolute Riesz summability of the type $\exp\{(\log w)^{\Delta}\}$ ($\Delta > 0$ however large) and order $1+\delta$ ($\delta > 0$) of Fourier series at a point $t = x$. Our second object is to obtain a criterion for the absolute convergence of Fourier series by using above mentioned absolute Riesz summability. Precisely we prove the following theorems :

Theorem 1—Let

$$k \geqslant \pi e^2 \text{ and } 0 < p < 1. \qquad \ldots \quad (2.1)$$

Then

$$\phi(t) \log\log \frac{k}{t} \in BV(0,p) \qquad \ldots \quad (2.2)$$

implies

$$\sum_{n=1}^{\infty} A_n(x) \in |R, \quad \exp((\log w)^{\Delta}), \quad 1+\delta|, \qquad \ldots \quad (2.3)$$

where $\delta > 0$, $\Delta > 0$ however large.

*For definitions see Chandra (1971*b*) and Mohanty (1951).

Theorem 2—Let (2.1) and (2.2) hold.

Then $$\sum_{n=1}^{\infty} A_n(x) \in |C, 0|,$$

if and only if,

$$\{A_n(x)n(\log n)^{1-\Delta}\} \in |R, \exp((\log w)^{\Delta}), 1| \quad \ldots (2.5)$$

where $\Delta > 0$ however large.

3. Lemmas

We require the following lemmas for the proof of the theorems :

Lemma 1—If $F(x) \in BV(a, b)$, then it can be expressed as $(F_1(x)-F_2(x))$ where $F_i(x)$ $(i = 1, 2)$ are positive, bounded and monotonic increasing function in (a, b) (see Carslaw 1930, p. 83).

Lemma 2—If $\sum_{n=1}^{\infty} a_n \in |R, \lambda_n, r| (r \geqslant 0)$, then $\sum_{n=1}^{\infty} a_n \in |R, \lambda_n, r'|$, for $r' > r$ (Obrechkoff 1928, 1929).

Lemma 3—If a series $\sum_{n=1}^{\infty} a_n \in |R, \lambda(w), r| (r > 0)$ and $\mu(w)$ is a logarithmico-exponential function of $\lambda(w)$ such that

$$\mu(w) = O\{(\lambda(w))^{\Delta}\}$$

where Δ is a constant, then the series $\sum_{n=1}^{\infty} a_n \in |R, \mu(w), r|$ (Chandrasekharan, 1942).

See Mohanty (1951) for $|R, \lambda_n, r| \sim |R, \lambda(w), r|$ $(r > 0)$.

Lemma 4—for any $\{\lambda_n\}, \{b_n\} \in |R', \lambda_n, 1|$ and $\{d_n\} \in BV$ imply $\{b_n d_n\} \in |R', \lambda_n, 1|$, where $(R', \lambda_n, 1)$ is the discrete Riesz mean of the type λ_n and order unity.

For its proof see Das (1969, Lemma 17).

This is explicitly proved by Das for $|R', \lambda_{n-1}, 1|$ and the same arguments hold good for $|R', \lambda_n, 1|$ which is equivalent to $|R, \lambda_n, 1|$ (see Mohanty 1951, footnote to the p. 298).

Lemma 5—If $\sum_{n=1}^{\infty} a_n \in |R, \exp((\log n)^{\Delta}), r|$, where $\Delta > 0$ and $r > 0$, then a necessary and sufficient condition for $\sum_{n=1}^{\infty} a_n \in |C, 0|$ is

$$\{na_n(\log n)^{1-\Delta}\} \in |R', \exp((\log n)^{\Delta}), 1| \quad \ldots (3.1)$$

Proof: *Condition necessary*—Let $s_n = a_1+a_2+\ldots+a_n$ and $\{s_n\} \in BV$. Then $\{s_n\} \in |R', \lambda_n, 1|$, which is equivalent to

$$\left\{ s_n - \frac{1}{\lambda_{n+1}} \sum_{m=1}^{n} \lambda_m a_m \right\} \in BV$$

and which further implies

$$\left\{ \frac{a_n \lambda_n}{\lambda_{n+1}-\lambda_n} \right\} \in |R', \lambda_n, 1|. \qquad \ldots (3.2)$$

Now, since for $\lambda_n = \exp((\log n)^{\Delta})$ $(\Delta > 0)$

$$\left\{ \frac{n(\lambda_{n+1}-\lambda_n)}{\lambda_n (\log n)^{-1+\Delta}} \right\} \in BV \qquad \ldots (3.3)$$

we follow the necessity part by using (3.2) and (3.3) in Lemma 4.

Condition sufficient—Since

$$\left\{ \frac{\lambda_n (\log n)^{-1+\Delta}}{n(\lambda_{n+1}-\lambda_n} \right\} \in BV, \quad (3.1) \text{ implies } (3.2),$$

that is,

$$\left\{ \frac{1}{\lambda_{n+1}} \sum_{m=1}^{n} \lambda_m a_m \right\} \in BV,$$

by Lemma 4, which further implies and is implied by

$$\text{(i)} \left\{ \frac{\lambda_n}{\lambda_{n+1}} \right\} \in BV \text{ and } \quad \text{(ii)} \left\{ \frac{1}{\lambda_n} \sum_{m=1}^{n} \lambda_m a_m \right\} \in BV.$$

Therefore (3.1) is sufficient since (i) and (ii) are sufficient conditions (see Bhatt 1958) for $\sum_{n=1}^{\infty} a_n \in |C, 0|$, whenever $\sum_{n=1}^{\infty} a_n \in |R, \lambda_n, r|$ $(r > 0)$.

This terminates the proof of the Lemma.

4. Proof of the theorems

We shall first prove Theorem 1 and then by using this we shall prove Theorem 2. In view of Lemma 3 we take $\Delta > 1$ for the proof of Theorem 1.

Proof of Theorem 1—Since

$$A_n(x) = \frac{2}{\pi} \int_0^{\pi} \phi(t) \cos nt \, dt,$$

the series $\sum_{n=1}^{\infty} A_n(x) \in |R, e(w), 1+\delta|(\delta > 0)$

if

$$I = \frac{2}{\pi} \int_e^\infty \frac{(\log w)^{\Delta-1}}{we^{1+\delta}(w)} \left| \int_0^\pi \phi(t)K(w, t)dt \right| dw$$

is convergent.

Now, for $0 < p < 1$, we write

$$I \leqslant \frac{2}{\pi} \int_e^\infty \frac{(\log w)^{\Delta-1}}{we^{1+\delta}(w)} \left| \int_0^p \phi(t)\, K(w, t)dt \right| dw$$

$$+\frac{2}{\pi} \int_e^\infty \frac{(\log w)^{\Delta-1}}{we^{1+\delta}(w)} \left| \int_p^\pi \phi(t)K(w, t)dt \right| dw$$

$$= I_1 + I_2, \quad \text{say.}$$

By using (1.7), the boundedness of I_2 can easily be established. Therefore, in view of Lemma 1 and the second mean value theorem, for the proof of the theorem, it is enough to show that, writing $\log_2$ for log log,

$$J_1 = \int_e^\infty \frac{(\log w)^{\Delta-1}}{we^{1+\delta}(w)} |E(w, t)| dw = O\left(\log_2 \frac{k}{t} \right) \quad \ldots \quad (4.1)$$

uniformly in $0 < t \leqslant 1$. And

$$J_2 = \int_e^\infty \frac{(\log w)^{\Delta-1}}{we^{1+\delta}(w)} \left| \sum_{n \leqslant w} (e(w) - e(n))^\delta e(n)\alpha_n \right| dw \quad \ldots \quad (4.2)$$

is convergent, where

$$\alpha_n = \frac{1}{n} \int_0^p \left(\frac{\sin nu}{u \log \frac{k}{u} \left(\log_2 \frac{k}{u} \right)^2} \right) du.$$

By moving parallel to (3.2) of Chandra (1970c) we have

$$\alpha_n = O\left\{ \frac{1}{n \log n (\log_2 n)^2} \right\}$$

Therefore, the convergence of J_2 follows by Lemma 2.

Proof of (4.1)—Let $T_1 = k/t$ and

$$T_2 = \frac{k}{t} \left\{ \log \frac{k}{t} \right\}^{\Delta-1}.$$

Then, we write

$$J_1 = \int_e^{T_1} + \int_{T_1}^{T_2} + \int_{T_2}^\infty$$

$$= J_{1,1} + J_{1,2} + J_{1,3}, \quad \text{say.}$$

By using (i) and (ii) of (1.6) respectively, we get

$$J_{1,1} = O(1),$$

and

$$J_{1,2} = O\{\log T_2 - \log T_1\} = O\left\{\log\log\frac{k}{t}\right\}.$$

And finally, by (1.6) (iii), we have

$$J_{1,3} = O\left\{t^{-1}\int_{T_2}^{\infty}\frac{(\log w)^{\Delta-1}}{w^2}\,dw\right\}$$

$$= O\left\{\frac{(\log T_2)^{\Delta-1}}{tT_2}\right\}$$

$$= O(1).$$

Combining the above results, we have

$$J_1 = O\left\{\log\log\frac{k}{t}\right\}.$$

This terminates the proof of the Theorem.

Proof of Theorem 2—For the proof of this theorem we use the standard definition of absolute Riesz summability.

Under the hypothesis of Theorem 2 we follow, by using Theorem 1, that $\sum_{n=1}^{\infty} A_n(x) \in |R, \exp((\log w)^{\Delta}), 1+\delta|$ ($\delta > 0$ and $\Delta > 0$) and hence by Lemma 5, $\sum_{n=1}^{\infty} A_n(x) \in |C, 0|$ if and only if (3.1), with $A_n(x)$ in place of a_n, holds. Again using $|R', \lambda_n, 1| \sim |R, \lambda_n, 1| \sim |R, \lambda(w), 1|$ (see Mohanty 1951, footnote to p. 298 and definition B), we follow the proof of the theorem.

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