NUMERICAL STUDY OF UNSTEADY STAGNATION-POINT FLOWS

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In this paper, the unsteady flow motion in the two-dimensional boundary layer of an incompressible fluid over the flat plate oriented normally to the oncoming flow is investigated. Uniform as well as non-uniform oncoming flows are considered. In all the cases the steady state function of the unsteady laminar boundary layer equations were obtaind. The results of numerical calculations are represented graphically.

1. INTRODUCTION

The method of finite difference approximation (FDM) in solving two-dimensional problems in fluid mechanics has been widely used in recent years. This technique has enabled to investigate some of those characteristics of the fluid flows which were not amenable to analytical or other methods.

Here an attempt is made to study the unsteady flow motion in the two-dimensional boundary layer of an incompressible fluid over the flat plate oriented normally to the oncoming ideal flow, i.e. the flow which is known as stagnation point flow. There are some schemes for the numerical calculation of unsteady boundary layer equations (Rosliakov and Chernov 1971, Farn and Arpaci 1966, Hall 1969), but to the authors concern, the steady state solution for the unsteady stagnation-point flow has not been obtained as yet, though this fact may appear strange in view of the many technical applications of these flow motions.

2. FORMULATION OF THE PROBLEM

Consider the two dimensional flow of an incompressible fluid impinging normally on a flat plate (plate in $X-Z$ plane, normal to the Y-direction). The oncoming flow is assumed to be inviscid and steady, the point of stagnation thus formed on the flat plate is taken as $X = Y = 0$. Let Y_{σ} denote the distance between the plate and the oncoming stream where the influence

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of the plate on the ideal fluid motion is negligibly small. We take the velocity of the oncoming flow at Y_∞ to be V_∞ and assume the fluid flow to be symmetrical about $X = 0$. The velocity components in the boundary layer are denoted by $\bar{u}$, $\bar{v}$ in the directions of X- and Y-axes; the corresponding velocity components of the oncoming stream are $\bar{U}$ and $\bar{V}$ respectively. We assume that the boundary layer formation takes place instantaneously from the ideal fluid motion at the initial instant.

Case I : Uniform Flow.

The solution of the steady stagnation-point flow known as the Hiementz solution (Schlichting 1961), is based on the linear distribution of the velocity components along the X- and Y-axes in the potential ideal flow. It is easy to show that the linear law of the velocity distribution holds good only for small distance from the surface of the plate.

Assuming $\bar{V}/V_\infty = -F(Y/Y_\infty)$, we obtain from the equation of continuity that the $\bar{U}$-component of velocity along the plate surface is $\bar{U} = F'(X/Y_\infty)$, where function F can be evaluated from the following equations of the vortex transport;

$$F'''F - F'F'' = 0. \qquad \dots \quad (1)$$

The boundary conditions are

$$Y/Y_\infty = 0, \quad F = 0; \; Y/Y_\infty = 1.0, \quad F = 1.0, \; F' = 0.$$

Solution is

$$F = \cos\frac{\pi}{2}\,(1 - Y/Y_\infty).$$

Substituting F in the equation for velocity components, we obtain;

$$\bar{V} = -V_\infty \cos\frac{\pi}{2}(1 - Y/Y_\infty\,); \; \bar{U} = \frac{\pi}{2}\,\frac{V_\infty}{Y_\infty}\,X \sin\frac{\pi}{2}\,(1 - Y/Y_\infty) \qquad \dots \quad (2)$$

When Y is very small, velocity components can be approximated in the form

$$\bar{V} = -\beta Y, \quad \bar{U} = \beta X \qquad \dots \quad (3)$$

where $\beta = \frac{1}{2}\pi V_\infty / Y_\infty$ is the velocity gradient at the point of stagnation.

Case II : Non-Uniform Flow.

Solution (2) may be further extended for a non-uniform oncoming flow for which the velocity is a function of X at the distance Y_∞ from the plate. Introducing

$$\frac{\bar{V}}{V_\infty} = -F(Y/Y_\infty)\phi'(X/Y_\infty)$$

the equation of continuity gives

$$\frac{\bar{U}}{V_\infty} = F'(Y/Y_\infty)\phi(X/Y_\infty).$$

Using these expressions for velocity components the vortex transport equation can be written as :

$$\frac{\phi'\phi''-\phi\phi'''}{\phi\phi'} = \frac{F'F''-FF'''}{FF'} = \text{constant} \qquad \ldots \quad (4)$$

If we assume the constant to be zero in (4) we obtain two equations of the type of (1) for the function ϕ and F. The boundary conditions are

$$\frac{Y}{Y_\infty} = 0, \quad F = 0; \quad \frac{Y}{Y_\infty} = 1, \quad F = 1, \; F' = 0$$

$$\frac{X}{Y_\infty} = 0, \quad \phi = 0; \quad \frac{X}{Y_\infty} = \frac{X_\infty}{Y_\infty}, \quad \phi = 1, \quad \phi' = 0.$$

Solution of eqn. (4) is given by

$$\bar{V} = -\frac{\pi}{2}\cdot\frac{Y_\infty}{X_\infty}\,.\,V_\infty\,.\cos\frac{\pi}{2}\,(1-Y/Y_\infty\,)\sin\frac{\pi}{2}\,(1-X/X_\infty\,) \quad \ldots \quad (5)$$

$$\bar{U} = \frac{\pi}{2}\,.\,V_\infty\,.\sin\frac{\pi}{2}\,(1-Y/Y_\infty)\,.\cos\frac{\pi}{2}\,.\,(1-X/X_\infty) \quad \ldots \quad (6)$$

In order to obtain the same values of the velocity gradient at the point of stagnation as provided by (3), we choose $X_\infty = \frac{1}{2}\pi\,.\,Y_\infty$.

3. Boundary Layer Equations and the Boundary Conditions

The system of equations describing the unsteady flow motions of an incompressible fluid in the two-dimensional boundary layer of the flat plate is as follows :

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0 \qquad \ldots \quad (7)$$

$$\frac{\partial u}{\partial t}+u\,\frac{\partial u}{\partial x}+v\,\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x}+\frac{\partial^2 u}{\partial y^2} \qquad \ldots \quad (8)$$

where

$$x = \frac{X}{Y_\infty}; \quad y = \frac{Y.\sqrt{Re}}{Y_\infty}; \quad u = \frac{\bar{u}}{V_\infty}; \quad v = \frac{\bar{v}}{V_\infty}\,.\sqrt{Re}$$

$$t = \frac{\bar{t}\,V_\infty}{Y_\infty}; \quad p = \frac{\bar{p}}{\rho V_\infty^2}; \qquad Re = \frac{V_\infty Y_\infty}{\mu}$$

t is the time and μ the kinematic viscosity.

Equations (7) and (8) are to be solved subject to the following boundary conditions :

$$t > 0;\ y = 0,\ u = 0,\ v = 0$$

$$y \to \infty,\ u \to U(x).$$

For numerical calculations we take ∞ as y_∞ which is chosen sufficiently large. At $t = 0$ we use the solutions obtained for the ideal stagnation point flow in order to satisfy the initial conditions in the whole domain of calculations ($0 \leqslant y < \infty$, $0 \leqslant x \leqslant x_\infty$, where $x_\infty = X_\infty / Y_\infty$ for a non-uniform oncoming flow), that is

Uniform flow : $u = U = \frac{\pi}{2} \cdot x \cos \frac{\pi}{2} \frac{y}{\sqrt{Re}}$

$$v = V = -\sqrt{Re} \sin \frac{\pi}{2} \frac{y}{\sqrt{Re}}$$

At higher Reynolds number and $y \ll 1.0$, we have

$$u \simeq \frac{\pi}{2} x,\ v \simeq -\frac{\pi}{2} y. \qquad \ldots \quad (10)$$

Non-uniform flow : $u = U = \frac{\pi}{2} \cos \frac{\pi}{2} \frac{y}{\sqrt{Re}} \sin x \qquad \ldots \quad (10a)$

$$v = V = -\sqrt{Re} \sin \frac{\pi}{2} \frac{y}{\sqrt{Re}} \cos x.$$

At any instant of time, we calculate v component of the velocity along the axis of symmetry ($x = 0$) by finding the solution for $u(t, x, y)$ (in the vicinity of the line of symmetry) from (Rosliakov and Chernov 1971) :

$$u = xu_1(t, x, y)$$

then

$$\frac{\partial u}{\partial x} = u_1 + x \cdot \frac{\partial u_1}{\partial x}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = u_1(t, 0, y).$$

Near the line of symmetry at $y = y_\infty$, we take

$$u_0 = xu_{10}(t, x) = U(x, y_\infty)$$

Then

$$\frac{\partial u_0}{\partial x} = \frac{\partial}{\partial x}[x \cdot u_{10}(t, x)] = x \frac{\partial u_{10}}{\partial x} + u_{10}$$

and $$\left.\frac{\partial u_0}{\partial x}\right|_{x=0} = u_{10}(t, 0).$$

Using Bernoulli's equation

$$-\frac{\partial p}{\partial x} = u_0 \cdot \frac{\partial u_0}{\partial x} + \frac{\partial u_0}{\partial t}$$

it follows that

$$-\frac{\partial p}{\partial x} = xu_{10} \cdot \left(x \frac{\partial u_{10}}{\partial x} + u_{10} \right) + x \frac{\partial u_{10}}{\partial t}$$

Hence, at $x = 0$

$$u_1 + \frac{\partial v}{\partial y} = 0 \qquad \dots \quad (11)$$

$$\frac{\partial u_1}{\partial t} + v. \frac{\partial u_1}{\partial y} = \frac{\partial^2 u_1}{\partial y^2} - u_1^2 + \frac{\partial u_{10}}{\partial t} + u_{10}^2. \qquad \dots \quad (12)$$

These equations are to be solved for u_1 and v as functions of t and y.

The boundary conditions are

$$y = 0,\ u_1 = 0,\ v = 0$$

$$y \to \infty,\ u_1 \to u_{10}$$

where $u_{10} = \dfrac{\partial U(x)}{\partial x}$ at the edge of the boundary layer, $y = y_\infty$.

4. Finite Difference method

We put up a mesh with mesh lengths $\Delta x = Q$, $\Delta y = H$ on the domain of the fluid flow. Let $G^m_{i,j}$ denote the value of the function G at the point $[x = (i-1)Q, y = (j-1)H]$ at time $t = m\Delta t$. Using forward differences for time derivatives and central differences for space derivatives, approximations for eqns. (7) and (8) can be written as:

Case I : Uniform Flow

$$\frac{u^{m+1}_{i,j} - u^m_{i,j}}{\Delta t} + u^m_{i,j}[u^m_{i+1,j} - u^m_{i-1,j}]/2Q + v^m_{i,j}[v^m_{i,j+1} - v^m_{i,j-1}]/2H$$

$$= -\frac{\partial p}{\partial x} + [u^m_{i,j+1} + u^m_{i,j-1} - 2u^m_{i,j}]/H^2 \qquad \dots \quad (13)$$

$$[u^{m+1}_{i,j-1} - u^{m+1}_{i-1,j-1} + u^{m+1}_{i,j} - u^{m+1}_{i-1,j}]/2Q + [v^{m+1}_{i-1,j} - v^{m+1}_{i-1,j-1} + v^{m+1}_{i,j} - v^{m+1}_{i,j-1}]/2H = 0. \qquad \dots \quad (14)$$

At the line of symmetry,

$$\frac{ul_j^{m+1}-ul_j^m}{\Delta t} = [ul_{j+1}^m+ul_{j-1}^m-2ul_j^m]/H^2-v_j^m[ul_{j+1}^m-ul_{j-1}^m]/2H$$

$$-[ul_j^m]^2+(\pi^2/4)\cos^2\frac{\pi}{2}\frac{y_\infty}{\sqrt{Re}} \quad \ldots \quad (15)$$

$$v_{j+1}^{m+1} = v_{j-1}^{m+1}-2H.ul_j^{m+1} \quad \ldots \quad (16)$$

where

$$\frac{\partial p}{\partial x} = -U\frac{\partial U}{\partial x} = -\frac{\pi^2}{4}(i-1)Q\cos^2\frac{\pi}{2}\frac{y_\infty}{\sqrt{Re}}$$

ul is the u-velocity gradient in x-direction, Δt is the time step.

Case II : Non-uniform Flow

The finite difference equations for this flow is the same as that of uniform flow. Here

$$\frac{\partial p}{\partial x} = -\frac{\pi^2}{8}\cdot\cos^2\frac{\pi}{2}\cdot\left(\frac{y_\infty}{\sqrt{Re}}\right)\sin 2(i-1)\cdot Q.$$

Boundary Conditions :

The boundary conditions and the initial conditions for the problem are as follows :

Case I : Uniform Flow

$$t=0;\ u_{i,j} = \frac{\pi}{2}\cdot(i-1)Q.\cos\left(\frac{\pi}{2}\frac{(j-1)H}{\sqrt{Re}}\right),$$

$$v_{i,j} = -\sqrt{Re}\sin\left(\frac{\pi}{2}\frac{(j-1)H}{\sqrt{Re}}\right)$$

$$t>0;\ y=0,\ u_{i,1}=0,\ v_{i,1}=0$$

$$y\to\infty,\ u_{i,\infty}\to U = \frac{\pi}{2}(i-1)Q.\cos\left(\frac{\pi}{2}\frac{y_\infty}{\sqrt{Re}}\right)$$

$$v_{i,\infty} = -\sqrt{Re}\,.\sin\left(\frac{\pi}{2}\frac{y_\infty}{\sqrt{Re}}\right)$$

$$x=0,\ u_{1,j}=0,\ v_{1,j}=v_j.$$

At the line of symmetry

$$t=0; \qquad ul_j = -[v_{j+1}-v_{j-1}]/2H$$

$$t>0; \qquad y=0,\ ul=0,\ v=0$$

$$y\to\infty,\ ul\to\frac{\pi}{2}\cos\frac{\pi}{2}\frac{y_\infty}{\sqrt{Re}}.$$

Case II : *Non-uniform Flow*

$t = 0;\qquad u_{i,j} = \frac{\pi}{2} . \cos \frac{\pi}{2} \frac{(j-1)H}{\sqrt{Re}} \sin (i-1)Q$

$$v_{i,j} = -\sqrt{Re} . \cos (i-1)Q . \sin \left(\frac{\pi}{2} \frac{(j-1)H}{\sqrt{Re}} \right)$$

$t > 0;\qquad y = 0;\quad u_{i,1} = 0,\ v_{i,1} = 0$

$$y \to \infty;\quad u_{i,\infty} \to U = \frac{\pi}{2} . \cos \frac{\pi}{2} \frac{y_\infty}{\sqrt{Re}} . \sin (i-1)Q$$

$$v_{i,\infty} = -\sqrt{Re} . \sin \frac{\pi}{2} \frac{y_\infty}{\sqrt{Re}} . \cos (i-1)Q$$

$$x = 0;\quad v_{1,j} = v_j,\ u_{1,j} = 0.$$

At the line of symmetry

$t = 0;\qquad ul_j = -(v_{j+1} - v_{j-1})/2H$

$t > 0;\qquad y = 0;\quad ul = 0,\ v = 0$

$$y \to \infty;\quad ul_\infty \to \frac{\pi}{2} . \cos \frac{\pi}{2} \frac{y_\infty}{\sqrt{Re}}.$$

We solve equations (13) and (14) according to the following steps :

(I) At time $t = 0$, we initialise by taking the ideal solution of the problem, satisfying the continuity as well as vortex transport equations.

(II) The v-component of velocity at the line of symmetry is obtained by solving the equations (15) and (16) by an iterative method at time Δt.

(III)The u-component of velocity at time Δt is obtained by the momentum equation (13).

(IV) Equation (14) is solved for the values of v by an iterative scheme. The iterative method is assumed to converge when the difference in the values of v satisfies the criterion

$$| v_{i,j}^{(k-1)} - v_{i,j}^{(k)} | < 10^{-6}$$

where the upper suffix in bracket denotes the number of iterative step.

(V) The u-velocity gradient ul at the line of symmetry is calculated at time Δt by the equation;

$$ul_j = -(v_{j+1} - v_{j-1})/2H.$$

(VI) We repeat steps (II), (III), (IV) and (V) at time $2\Delta t$. We repeat the cycle of six steps at later times, till the steady state is reached.

5. Stability Criterion

As is well known, the stability of a finite difference equation requires the boundedness of the finite difference solution as Δt approaches zero, whereas Δx and Δy are fixed (Carnahan *et al.* (1965)). Therefore, for a finite time domain T, $0 \leqslant m\Delta t \leqslant T$, where m may take any integer between 0 and ∞ as Δt tends to zero, the stability of eqns. (13) and (14) is

$$|u^{m+1}_{i,j}| \leqslant A_1, \quad |v^{m+1}_{i,j}| \leqslant A_2 \text{ provided } A_1 \text{ and } A_2$$

are two positive constants. As it is shown by Farn (1965), the sufficient stability conditions for eqns. (13) and (14) are given by the following relations :

$$1/(u^{m}_{i,j}/(\Delta x)+2/(\Delta y)^2) \geqslant \Delta t, \quad 2/|v_{i,j}| \geqslant \Delta y. \qquad \ldots \quad (17)$$

we have chosen the value of Δt satisfying this criterion.

6. Discussion

For numerical computation, we have chosen $Re = 10000$ and $Re = 100$, $y_\infty = 10$; $x_\infty = 1.0$ (uniform flow); $x_\infty = \frac{1}{2}\pi$ (non-uniform flow). For uniform

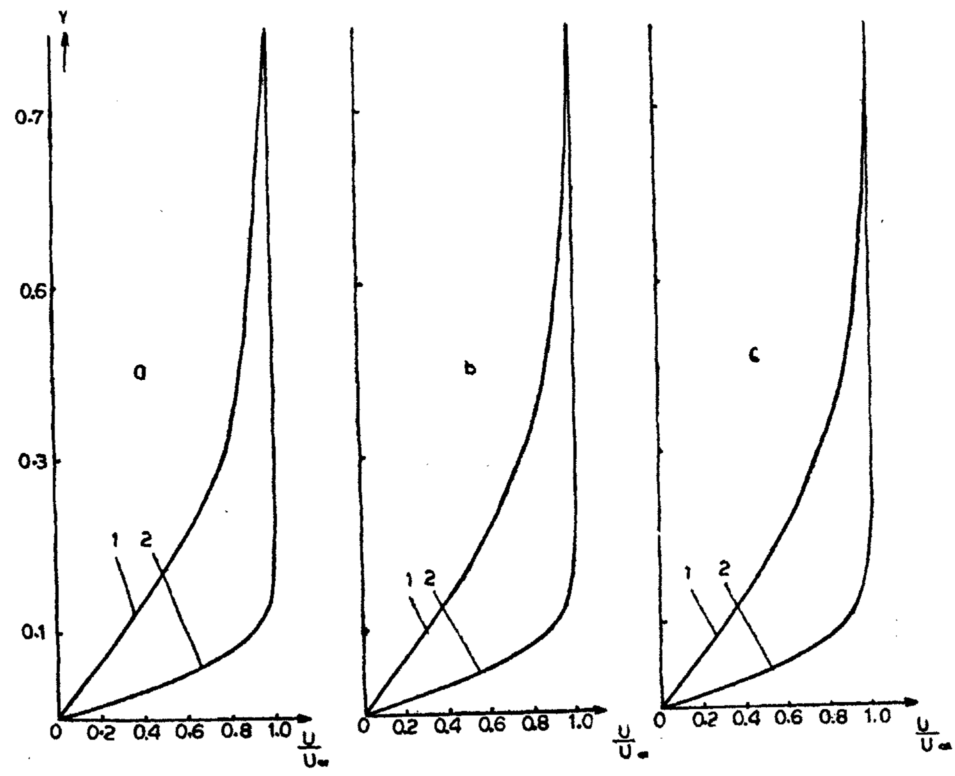

Fig. 1. u-velocity profile in the boundary layer for the uniform flow at $Re = 10^4$.
(a) $x = 0.2$; (b) $x = 0.6$; (c) $x = 1.0$.
1, $t = 0.032$; 2, $t = 0.002$

flow we have taken $\Delta x = \Delta y = 0.05$, and $\Delta x = \pi/40$, $\Delta y = 0.05$ for the non-uniform flow. For the uniform flow we take $\Delta t = 0.005$ and after four time steps, it is chosen as 0.001, for non-uniform flow we have taken $\Delta t = 0.001$. The final time of calculation was 0.032 for uniform flow. For non-uniform flow, the final time of calculation was 0.028 ($Re = 10^2$) and 0.032 ($Re = 10^4$). The numerical results at the final time of calculations indicate the existence of the steady state solutions.

The numerical calculations were done on the high speed digital computer CDC 3600 of Tata Institute of Fundamental Research, Bombay. Machine time required to complete computations for all the cases was about one hour.

The results thus obtained were used for plotting the following graphs represented by Figs. 1-9.

(i) $u = f(y)$, $u = f_1(y)$ against x at time $t = 0.002$ and $t =$ final.

(ii) Shear stress distribution on the wall $\tau_w = \left.\frac{\partial u}{\partial y}\right|_{y=0} = \psi(x)$ against x for several values of t.

The boundary layer velocity profiles (Figs. 1, 2) in the impingement region of a uniform flow at $Re = 10^4$ is very close to that of steady stagnation-point

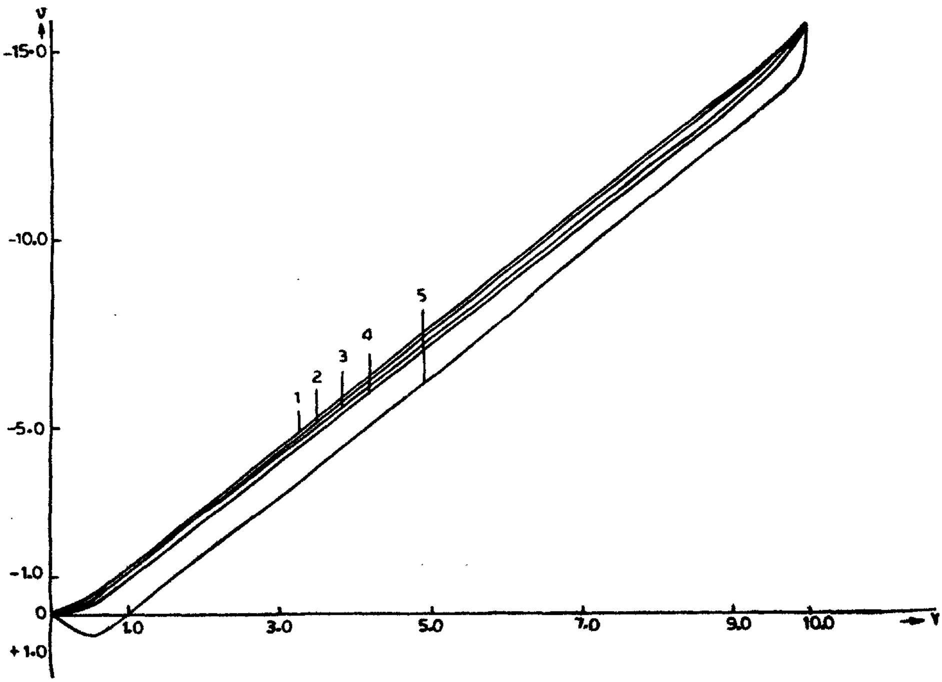

Fig. 2. v-velocity profile for uniform flow at $Re = 10^4$,
1, $x = 0$; 2, $x = 0.4$; 3, $x = 0.6$; 4, $x = 0.8$; 5, $x = 1.0$.

flow (Schlichting 1961); u-velocity profiles for both the cases are quite satisfactory coincedent for large values of x with difference within 10 per cent (Fig. 1). The process of forming the boundary layer is rather fast; the boundary layer grows rapidly near the stagnation-point. The behaviour of the normal (v) velocity component at different stations from the axis of symmetry is markedly different from that of the steady flow (Fig. 2). At small values of x, there is a reversal of the flow at the initial moment; it is reduced to the zone of very small

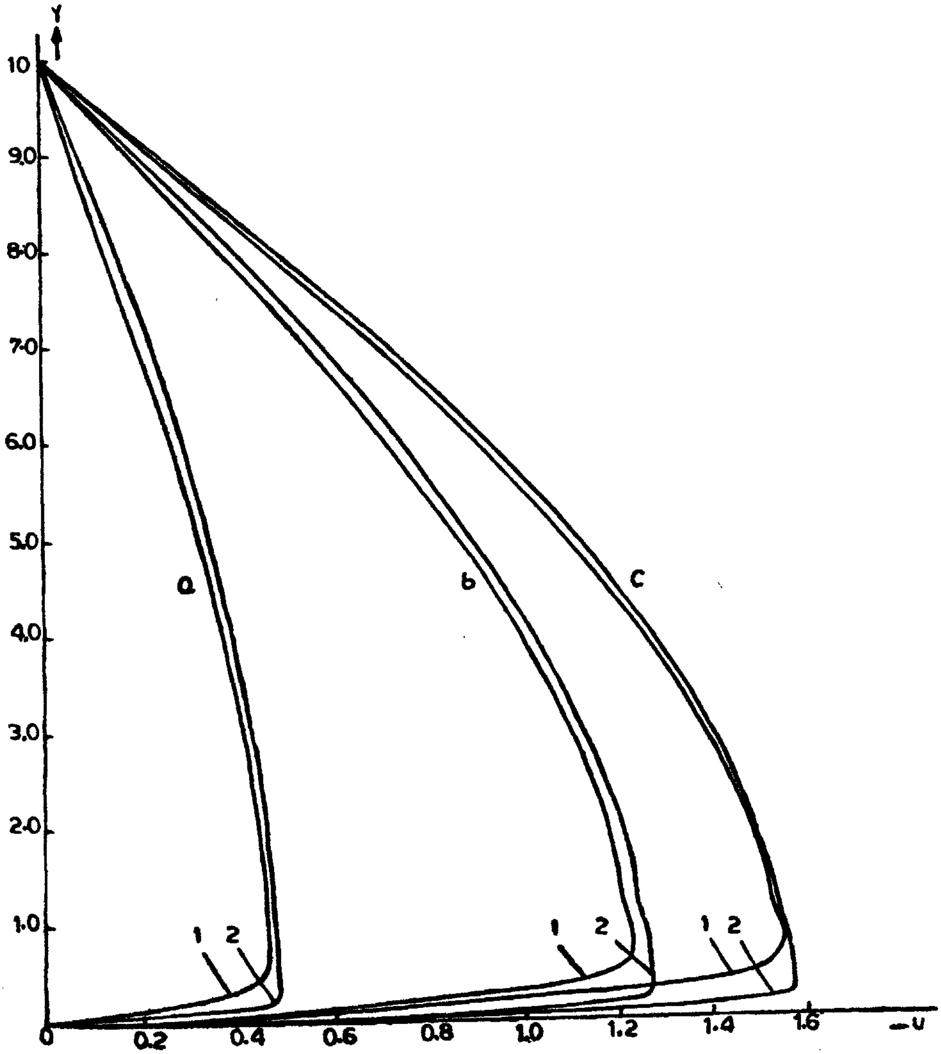

Fig. 3. u-velocity profile for the non-uniform flow at $Re = 10^2$.

(a) $x = \frac{\pi}{10}$; (b) $x = \frac{3\pi}{10}$; (c) $x = \frac{\pi}{2}$

1, $t = 0.028$; 2, $t = 0.004$

velocities near the surface of the plate as time progresses. At large values of x, the behaviour of v-component is close to that of steady stagnation point flow.

The velocity field in the impingement region of a non-uniform flow upon the flat plate is illustrated by Figs. 3 and 4 ($Re = 10^2$) and by Figs. 5 and 6 ($Re = 10^4$). As distinct from the uniform flow, the non-uniform character of the oncoming flow predominates the variation of the u, v components in the whole area of calculations. If at high values of Re, the u-velocity profiles resemble those of the steady stagnation point flow (Fig. 5), then for small

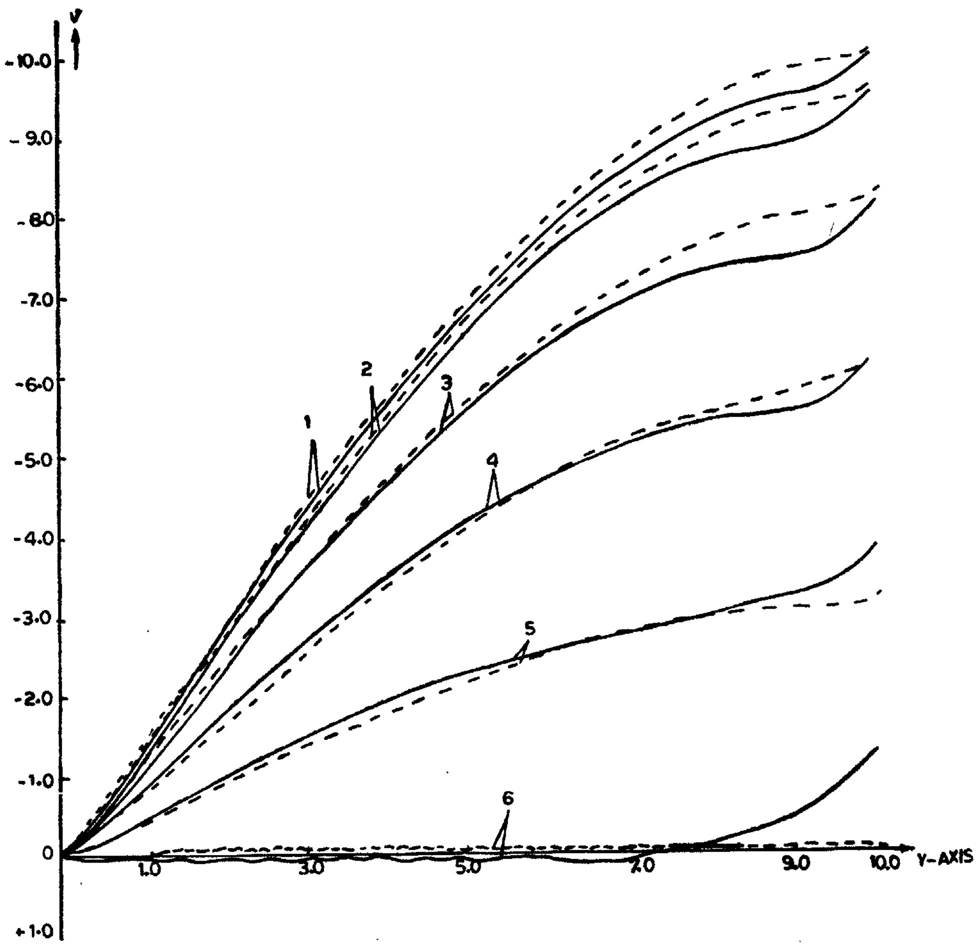

Fig. 4. v-velocity profile for the non-uniform flow at $Re = 10^2$, at $t = 0.004$(.........), at $t = 0.028$(————————) 1, $x = 0$; 2, $x = \pi/10$; 3, $x = \pi/5$; 4, $x = 3\pi/10$; 5, $x = 2\pi/5$; 6, $x = \pi/2$.

values of *Re* (Fig. 3) the difference is very large. In this case, the u-velocity profile in the boundary layer resembles that of a wall jet type. On account of a decrease in the outer velocity gradient in x-direction at high values of x, it is found that the boundary layer grows more rapidly compared to the previous case of the uniform flow. At high values of x, there is even a change in sign of the v-component (Figs. 4, 6) of velocity.

The wall shear stress increases linearly in conformity with the predictions of the steady stagnation-point flow for the uniform case (Fig. 7). In case of the non-uniform flows, the shear stress on the wall is of an exponential character at large values of x (Figs. 8, 9), this effect is in agreement with the results of calculations of the wall shear stress obtained by Belov and Pamadi (1972). In both the cases, the wall shear stress distribution shows a remarkable tendency towards the steady state solution as time increases.

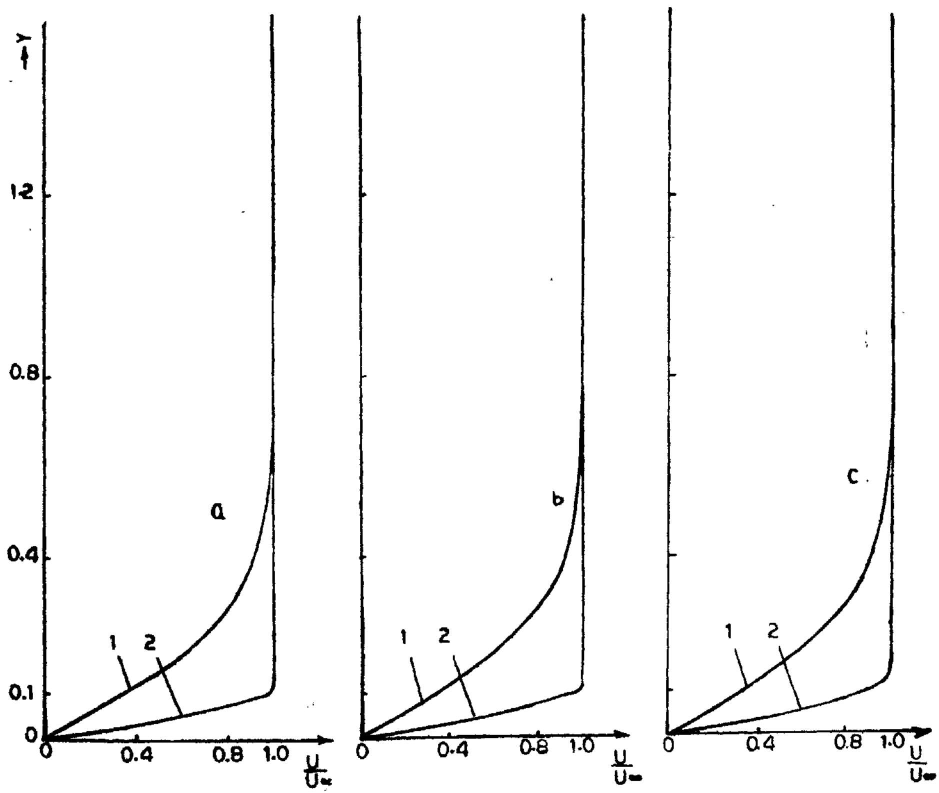

Fig. 5. u-velocity profile for the non-uniform flow at $Re = 10^4$.
(a) $x = \pi/10$; (b) $x = 3\pi/10$; (c) $x = \pi/2$
1, $t = 0, 032$; 2, $t = 0.002$.

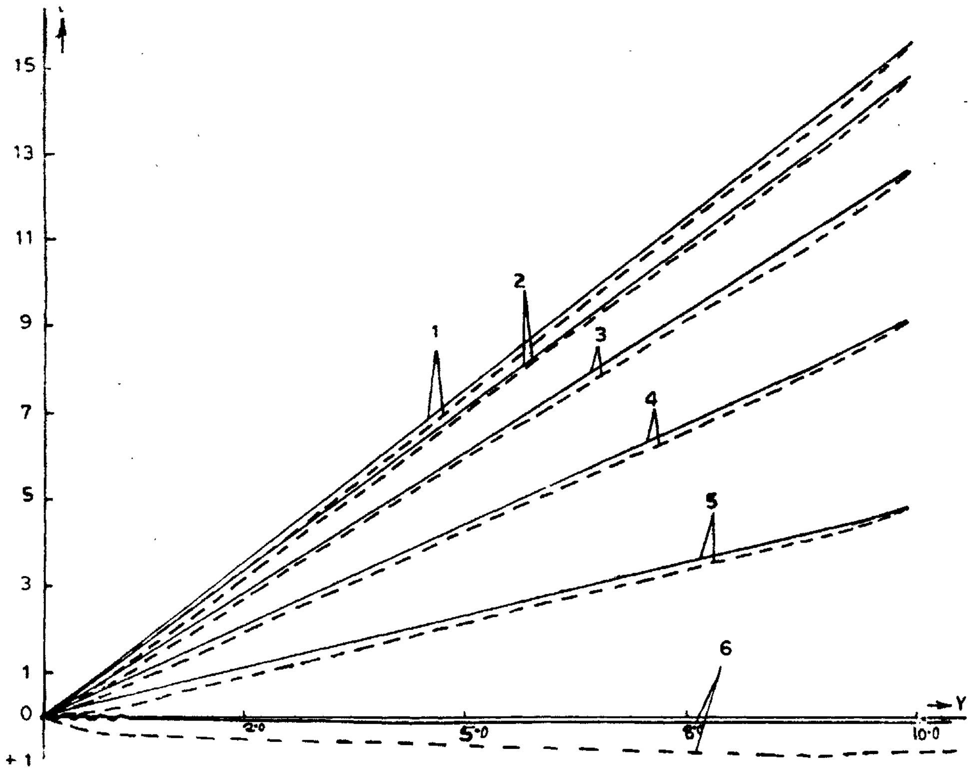

Fig. 6. v-velocity profile for the non-uniform flow at $Re = 10^4$, at $t = 0.002$(————), at $t = 0.032$ (– – – –)
1, $x = 0$; 2, $x = \pi/10$; 3, $x = \pi/5$; 4, $x = 3\pi/10$; 5, $x = 2\pi/5$; 6, $x = \pi/2$.

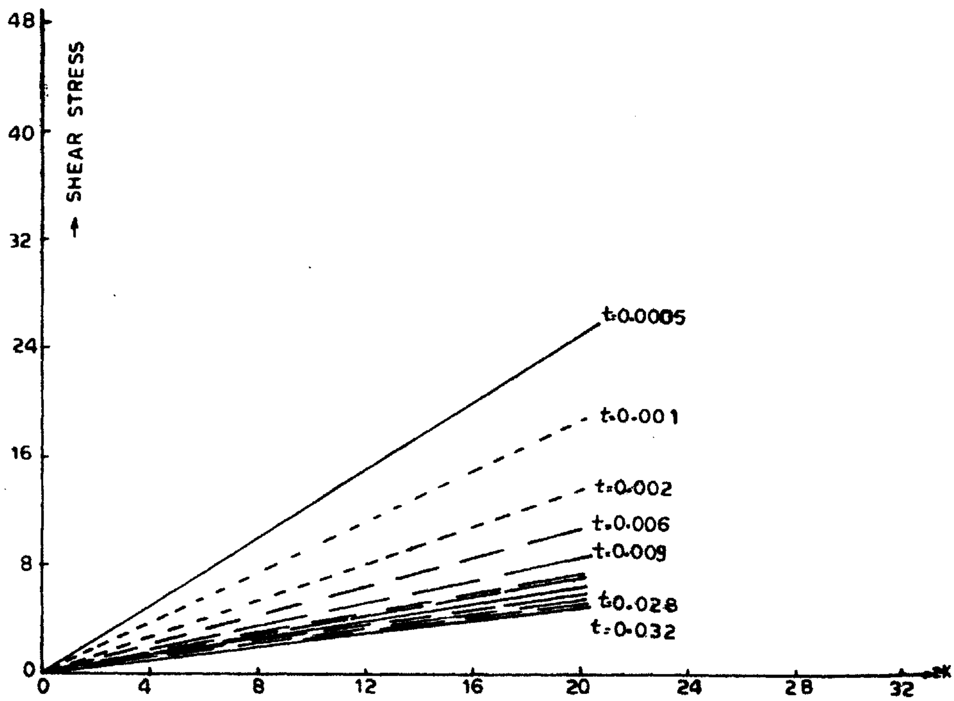

Fig. 7. Wall shear stress distribution for the uniform flow at different time t.

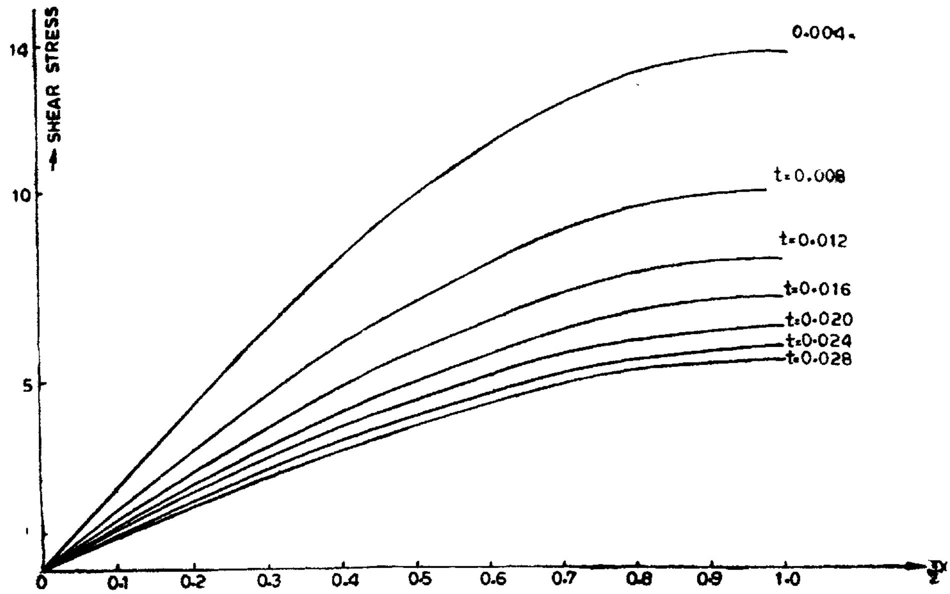

Fig. 8. Wall shear stress distribution for the non-uniform flow at $Re = 10^2$, at different time t.

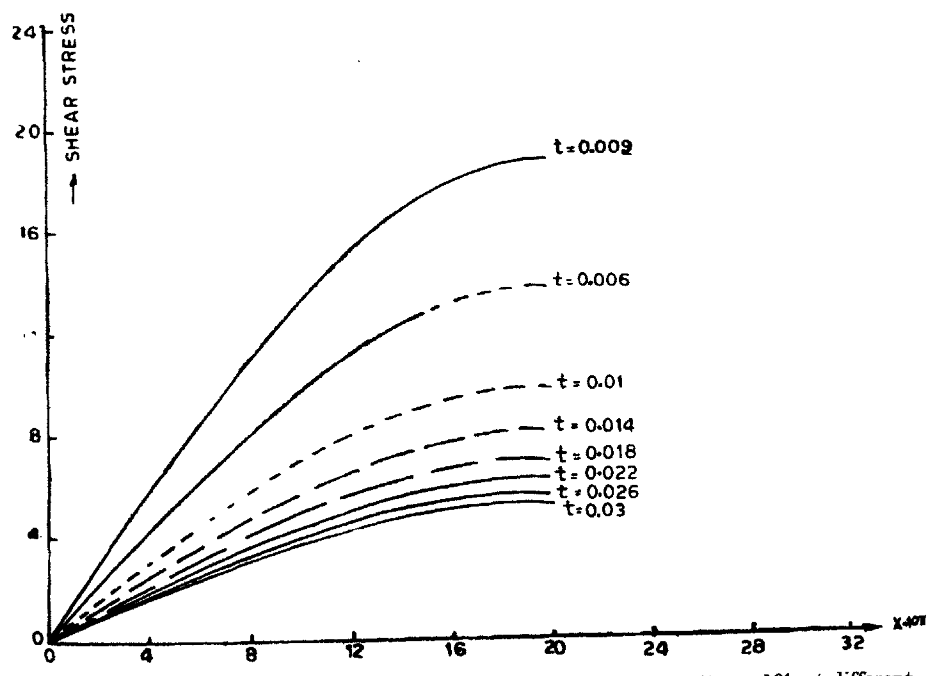

Fig. 9. Wall shear stress distribution for the non-uniform flow at $Re = 10^4$, at different time t.

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