SOLUTION OF STRESSES IN A SPHERE DUE TO A UNIFORM HEAT SOURCE FOR STRAIN HARDENING MATERIAL BY PERTURBATION METHOD

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The stresses in a sphere due to a uniform heat source is found for variety of representative materials. The solutions are obtained by perturbation method based on the elastic solutions and compared with the solutions obtained numerically with the help of digital computer.

INTRODUCTION

There are solutions for stresses in a sphere due to uniform heat source for elastic perfectly plastic material. This can be found in Johnson and Mellor's book (1962) and also further details concerning this problem will be found in the book by Glassstone (1956). Ghosh Dastidar and Ghosh (1972) obtained stresses numerically with the help of IBM digital computer for variety of representative materials using the Ramberg-Osgood (1943) uniaxial stress-strain relation. In this paper the same problem has been considered but the solutions are obtained by perturbation method following the procedure given by Tuba (1968). The perturbation method has been taken up since in this case the solutions can be written out for the complete distribution of stresses in equation form, similar to closed-form solutions. This is an advantage over the numerical solution.

GOVERNING EQUATIONS

In this case the equation of equilibrium takes the form

$$\frac{d\sigma_r}{dr}+\frac{2(\sigma_r-\sigma_\theta)}{r}=0 \qquad \ldots \quad \ldots \quad \ldots \quad (1)$$

If u be the radial displacement, the radial and tangential strains are given by

$$\epsilon_r = \frac{du}{dr} \qquad \ldots \quad \ldots \quad \ldots \quad (2)$$

$$\epsilon_\theta = \frac{u}{r}$$

which must satisfy the compatibility equation

$$\epsilon_r = \frac{d}{dr}(r\epsilon_\theta). \qquad \ldots \quad \ldots \quad \ldots \quad (3)$$

The total strain at any point is the sum of that due to stress and that due to temperature rise T, thus the strains are related to the stresses by

$$\epsilon_r = \frac{1}{E_s}(\sigma_r - \sigma_\theta) + \alpha T$$

$$\epsilon_\theta = \frac{1}{2E_s}(\sigma_\theta - \sigma_r) + \alpha T \qquad \ldots \quad \ldots \quad \ldots \quad (4)$$

where α is the coefficient of linear thermal expansion. It is assumed here that α is unaffected by temperature.

The secant modulus E_s depends on the effective stress

$$\bar{\sigma} = \sigma_\theta - \sigma_r \qquad \ldots \quad \ldots \quad \ldots \quad (5)$$

and is given by

$$\frac{1}{E_s} = \frac{1}{E}\left[1 + \frac{3}{7}\left(\frac{\bar{\sigma}}{\sigma_1}\right)^{n-1}\right] \qquad \ldots \quad \ldots \quad \ldots \quad (6)$$

where σ_1 is a nominal yield stress. Here E, the elastic modulus, is also unaffected by temperature.

The formula (6) has been derived from the Ramberg-Osgood equation for the uniaxial stress-strain curve

$$\epsilon = \frac{\sigma}{E}\left[1 + \frac{3}{7} \cdot \left(\frac{\sigma}{\sigma_1}\right)^{n-1}\right] \qquad \ldots \quad \ldots \quad \ldots \quad (7)$$

where ϵ is strain and σ stress. These stress strain curves for various values of n are shown in Fig. 1. For $n \to \infty$ the curve approached the elastic-perfectly plastic law.

It is assumed that the rate of generation of heat is a fixed amount per unit volume, Q and the temperature on the outside surface of the solid sphere of radius a is zero, then the temperature distribution is given by

$$T = Q.\frac{(a^2 - r^2)}{6K} \qquad \ldots \quad \ldots \quad \ldots \quad (8)$$

where K is the thermal conductivity.

The boundary conditions are

$$\sigma_r = 0 \quad \text{at } r = a \qquad \ldots \quad \ldots \quad \ldots \quad (9a)$$

$$\sigma_r = \sigma_\theta \quad \text{at } r = 0. \qquad \ldots \quad \ldots \quad \ldots \quad (9b)$$

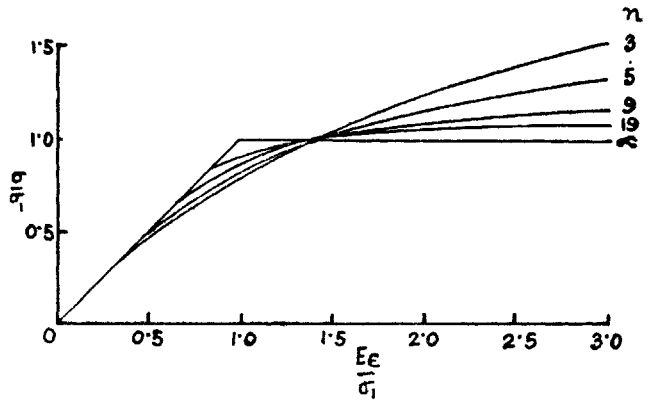


FIG. 1. Non-dimensional Ramberg-Osgood Stress-Strain Curves

SOLUTION BY PERTURBATION METHOD

Introducing a parameter μ as

$$\mu = \frac{3}{7} \cdot \frac{1}{\sigma_1^{n-1}} \qquad \ldots \quad \ldots \quad \ldots \quad (10)$$

and expressing the stresses in terms of the stress function F, i.e.,

$$\sigma_r = \frac{2}{r^2}\frac{dF}{dr}, \quad \sigma_\theta = \frac{1}{r}\frac{d^2F}{dr^2} \qquad \ldots \quad \ldots \quad \ldots \quad (11)$$

eqns. (6), (4) and the compatibility eqn. (3) become

$$\frac{1}{E_s} = \frac{1}{E}[1+\phi] \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

$$\epsilon_r = \frac{1}{E}(1+\phi)\left(\frac{2}{r^2}\frac{dF}{dr} - \frac{1}{r}\frac{d^2F}{dr^2}\right) + \frac{\alpha Q(a^2-r^2)}{6K}$$

$$\epsilon_\theta = \frac{1}{2E}(1+\phi)\left(\frac{1}{r}\frac{d^2F}{dr^2} - \frac{2}{r^2}\frac{dF}{dr}\right) + \frac{\alpha Q(a^2-r^2)}{6K} \qquad \ldots \quad (13)$$

$$\left(\frac{1}{r}\cdot\frac{d^3F}{dr^3} - \frac{2}{r^3}\cdot\frac{dF}{dr} - \frac{2E\alpha Q r}{3K}\right) + \frac{d\phi}{dr}\left(\frac{1}{r}\frac{d^2F}{dr^2} - \frac{2}{r^2}\frac{dF}{dr}\right)$$

$$+\phi\left(\frac{1}{r}\frac{d^3F}{dr^3} - \frac{2}{r^3}\frac{dF}{dr}\right) = 0 \qquad \ldots \quad \ldots \quad \ldots \quad (14)$$

where

$$\phi = \mu\left(\frac{1}{r}\frac{d^2F}{dr^2} - \frac{2}{r^2}\frac{dF}{dr}\right)^{n-1}. \qquad \ldots \quad \ldots \quad (15)$$

Next assumption is that the stress function F can be expanded in Taylor series form

$$F = \mu^i F_i = \sum_{i=0,\,1,\,\ldots} \mu^i F_i. \qquad \ldots \quad \ldots \quad (16)$$

The expansion is given for the neighbourhood of $\mu = 0$ and this corresponds to the elastic solution.

Now for $n = 3$ the governing eqn. (14) with the help of eqns. (15) and (16) become.

$$\mu^m\left[\frac{1}{r}\frac{d}{dr}\left\{r^4\frac{d}{dr}(F_m/r^2)\right\}\right] - \frac{2E\alpha Qr}{3K}$$

$$= -\mu^{i+j+k+1}\left[\frac{2}{r^3}\dot{F}_i\ddot{F}_j\ddot{F}_k - \frac{6}{r^4}\ddot{F}_i\ddot{F}_j\ddot{F}_k + \frac{20}{r^5}\dot{F}_i\ddot{F}_j\ddot{F}_k - \frac{16}{r^6}\dot{F}_i\dot{F}_j\ddot{F}_k\right.$$

$$- \frac{4}{r^4}\dot{F}_i\dddot{F}_j\ddot{F}_k - \frac{4}{r^4}\ddot{F}_i\dddot{F}_j\dot{F}_k + \frac{10}{r^5}\ddot{F}_i\ddot{F}_j\dot{F}_k - \frac{32}{r^6}\dot{F}_i\ddot{F}_j\dot{F}_k + \frac{24}{r^7}\dot{F}_i\dot{F}_j\dot{F}_k$$

$$\left. + \frac{8}{r^5}\dot{F}_i\dddot{F}_j\dot{F}_k + \frac{1}{r^3}\ddot{F}_i\ddot{F}_j\dddot{F}_k + \frac{4}{r^5}\dot{F}_i\dot{F}_j\dddot{F}_k - \frac{4}{r^4}\dot{F}_i\ddot{F}_j\dddot{F}_k\right] \qquad \ldots \quad (17)$$

where dot indicates derivations of F with respect to r and i, j, k and m are dummy variables.

In eqn. (17), coefficients of the like powers of μ of the two sides must be equal. Now for $m = 0$ which gives the elastic solution, the eqn. (17) becomes

$$\frac{1}{r^3}\cdot\frac{d}{dr}\left[r^4\cdot\frac{d}{dr}\left(\frac{\dot{F}_0}{r^2}\right)\right] = \frac{2E\alpha Qr}{3k}. \qquad \ldots \quad \ldots \quad (18)$$

The solution of the differential equation (18) satisfying the boundary conditions (9*a*) and (9*b*) is given by

$$F_0 = \frac{2E\alpha Q}{3k}\cdot r^3\left[\frac{r^2}{5} - \frac{a^2}{3}\right]. \qquad \ldots \quad \ldots \quad \ldots \quad (19)$$

The process is carried out upto $m = 8$. For $m = 8$ there are 36 different combinations of i, j and k for which the right hand side is computed and summed up. The general expression F_m can be written as

$$F_m = \frac{Am.\,r^3}{(4m+3)(4m+2)}\cdot\left(\frac{r^{4m+2}}{4m+5} - \frac{a^{4m+2}}{3}\right) \qquad \ldots \quad (20)$$

where

$$A_m = (-1)^m . A_{m,3} . \left(\frac{2E\alpha Q}{15k} \right)^{2m+1} \quad \ldots \quad \ldots \quad \ldots \quad (21)$$

where the coefficients $A_{m,3}$ for different values of m are given in Table I.

Proceeding similarly as in the case $n = 3$ the expressions for F_m for $n = 5$ and $n = 9$ can be found out easily.

Table I

$A_{i\,n}$

i \ n	3	5	9
0	5	5	5
1	9	13	21
2	39	105	333
3	204	1015	4770
4	1155	10545	
5	6825		
6	41412		
7	255816		
8	847818		

With the help of the equation (16) and expressions for F for different values of n, F can be written as

$$F = \sum_{i=0,\,1,\,\ldots} \mu^i (-1)^i . A_{i,\,n} . \left(\frac{2E\alpha Q}{15k} \right)^{(n-1)i+1}$$

$$\times \frac{r^3}{\{2(n-1)i+5\}\{2(n-1)i+2\}} \left[\frac{r^{\{2(n-1)i+2\}}}{2(n-1)i+5} - \frac{a^{\{2(n-1)i+2\}}}{3} \right] \quad \ldots \quad (22)$$

where the values of $A_{i,\,n}$ for different values of i and n are given in Table I.

Now introducing the non-dimensional quantities

$$\zeta = \frac{r}{a}, \quad s_r = \frac{\sigma_r}{\beta}, \quad s_\theta = \frac{\sigma_\theta}{\beta} \quad \text{and} \quad \bar{s} = \frac{\bar{\sigma}}{\beta} = s_\theta - s_r. \quad \ldots \quad (23)$$

where

$$\beta = \frac{\alpha E Q a^2}{K} \quad \ldots \quad \ldots \quad \ldots \quad (24)$$

and with the help of eqn. (22), stresses can be found from eqn. (11) for different values of n as

$$s_\theta = \frac{\sigma_\theta}{\beta} = \sum_{i=0,1,..} (-1)^i \cdot \left(\frac{2}{15}\right) \cdot \left(\frac{3}{7}\right)^i \cdot \left(\frac{2\lambda}{15}\right)^{(n-1)i} \cdot (2A_{i,n}) \cdot \left[\frac{\{(n-1)i+2\}\zeta^{2(n-1)i+2}-1}{\{2(n-1)i+5\}\{2(n-1)i+2\}}\right]$$

and

$$s_r = \frac{\sigma_r}{\beta} = \sum_{i=0,1,..} (-1)^i \cdot \left(\frac{2}{15}\right) \cdot \left(\frac{3}{7}\right)^i \cdot \left(\frac{2\lambda}{15}\right)^{(n-1)i} (2A_{i,n}) \cdot \left[\frac{\zeta^{2(n-1)i+2}-1}{\{2(n-1)i+5\}\{2(n-1)i+2\}}\right] \quad \ldots \quad (25)$$

where

$$\lambda = \frac{\beta}{\sigma_1} \quad \ldots \quad \ldots \quad (26)$$

and the coefficients $A_{i,n}$ for different values of i and n are given in Table I. It is clear from eqn. (25) that the stresses are expressed in terms of ζ which is an advantage over solutions by numerical method.

There are certain limitations of this method. Equations (25) with the given number of terms are valid up to $\lambda \approx 5$, 5·5 and 6 for $n = 3$, 5 and 9 respectively. For values of λ larger than these, the results diverge.

This present solution can be extend to higher values of λ by applying Euler's transformation to the resulting series of eqns. (25). With the help of the Euler's transformation the transformed results are

$$s_\theta = \frac{2}{15}(2\zeta^2-1) + \sum_{i=1,2,..} (-1)^i \cdot B_{i,n} \cdot \left(\frac{\lambda}{10}\right)^{(n-1)i} [\{(n-1)i+2\}\zeta^{2(n-1)i+2}-1)]$$

and $\quad \ldots \quad (27)$

$$s_r = \frac{2}{15}(\zeta^2-1) + \sum_{i=1,2,..} (-1)^i \cdot B_{i,n} \cdot \left(\frac{\lambda}{10}\right)^{(n-1)i} [\zeta^{2(n-1)i+2}-1]$$

where the coefficients $B_{i,n}$ for different values of i and n are given in Table II.

Table II

$B_{i,n}$

i \ n	3	5	9
1	0·24807433	0·23809524	0·27881929
2	0·28100323	0·48175832	0·60637295
3	0·29715880	0·65337004	0·06895121
4	0·13832328	0·24981947	
5	$0{\cdot}14645778 \times 10^{-1}$		
6	$0{\cdot}19586116 \times 10^{-3}$		
7	$0{\cdot}14541993 \times 10^{-6}$		
8	$0{\cdot}54377400 \times 10^{-12}$		

The eqns. (27) are valid for $\lambda = 8$ for all values of n.

Figs. 2, 3 and 4 show the variation of s_r and s_θ with respect to ζ i.e., r/a for different values of n for $\lambda = 0{\cdot}4$, 5 and 8 respectively. For $\lambda = 0{\cdot}4$ and 5 results obtained by perturbation method and numerical method are so close that they are represented by the same curve. In Fig. 4 dotted lines show the results obtained by numerical method for $\lambda = 8$.

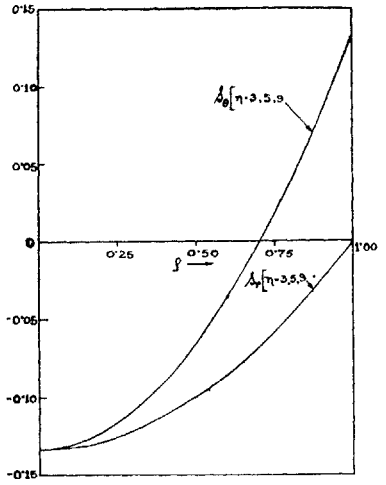


FIG. 2. Stress Distribution for $\lambda = 0{\cdot}4$

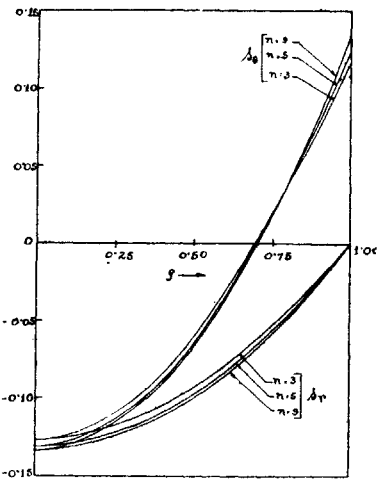
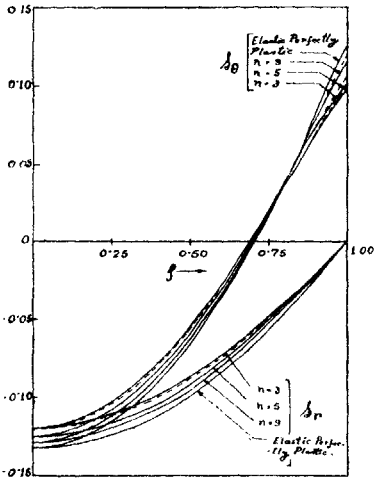


FIG. 3. Stress Distribution for $\lambda = 5{\cdot}0$



FIG, 4. Stress Distribution for $\lambda = 8$.

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