A NUMERICAL SOLUTION OF AN INTEGRAL EQUATION OF THE SECOND KIND OCCURRING IN AERODYNAMICS

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The paper describes a method for the numerical solution of a certain type of integral equations of the second kind. Numerical results are given for an example from aerodynamics.

1. Introduction

The theory of aerodynamics gives rise to integral equations of the second kind in the following form:

$$f(s)=F(s)+\int_0^1 K(s,t)\,f(t)\,dt,\ 0\leqslant s\leqslant 1 \qquad (1)$$

where the kernel $K(s,t)$ has the form,

$$K(s,t)=P(s,t)\log|s-t|+Q(s,t) \qquad (2)$$

$P(s,t)$, $Q(s,t)$ and $F(s)$ being continuous functions.

In § 2, a simple numerical method of solution of (1), based on the generalized trapezoidal rule, will be given and § 3 describes an error analysis for this method. In § 4, the method will be applied to an integral equation occurring in aerodynamics and finally, § 5 contains a discussion of results.

2. Numerical Method

The method of approximating the integral term of (1) is simple and is described here briefly. Details of derivation and computation are given by Sastry (1972). The integral equation (1) with the kernel (2), may be written as,

$$f(s)-\sum_{j=0}^{n-1}\int_{jh}^{\overline{j+1}h}\Big[P(s,t)\,log\,|s-t|+Q(s,t)\Big]f(t)\,dt = F(s).$$

Setting $s=ih$ and $t=jh+ph$, this gives,

$$f(ih)-h\sum_{j=0}^{n-1}\int_0^1\Big[P(ih,jh+ph)\,log\,|\overline{i-j-p}h|+Q(ih,jh+ph)\Big]$$

$$f(jh+ph)\,dp=F(ih).$$

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Applying now the generalized trapezoidal rule and using the notation $f_i = f(ih)$, we obtain,

$$\sum_{j=0}^{n-1} \int_0^1 \Big[\Big\{ (1-p)\, P_{ij}\, f_j + p P_{i,\,j+1}\, f_{j+1} \Big\} \log \mid \overline{i-j-ph} \mid dp$$

$$+ \{ (1-p)\, Q_{ij} f_j + p\, Q_{i,\,j+1} f_{j+1} \} \Big] \, dp - f_i / h = -F_i / h.$$

$$i = 1, 2, 3, \ldots, n-1. \qquad (3)$$

For the example to be considered in this paper, it was found that $f_0 = f_n = 0$; so that, in this case, (3) may be reduced to a system of $(n-1)$ linear equations in $(n-1)$ unknowns $f_1, f_2, \ldots, f_{n-1}$.

3. An Error Analysis

Consider the integral operator K, such that,

$$(\mathcal{K} f)(s) = \int_0^1 K(s, \sigma)\, f(\sigma)\, d\sigma \;, 0, \leqslant s \leqslant 1, \qquad (4)$$

where $f \in c[0,1]$.

If we define the maximum norm,

$$\| f \| = \max_{0 \leqslant s \leqslant 1} \mid f(s) \mid \qquad \ldots \qquad (5)$$

then the set c will be a Banach space. We adopt here the general theory of error analysis due to Anselone and Moore (1964), and Atkinson (1967).

Let $M = \max \mid \mathcal{K}(s, \sigma) \mid$ for $0 \leqslant x, y \leqslant 1$. Thus $\mathcal{K}$ is bounded and,

$$\| \mathcal{K} \| = \sup_{0 \leqslant s \leqslant 1} \int_0^1 \mid K(s, \sigma) \mid d\,\sigma \leqslant M\,. \qquad (6)$$

Since the set $\{ \mathcal{K} f : \| f \| \leqslant 1 \}$ is bounded and equicontinuous, it has compact closure by Arzela's theorem (Collatz 1966). Hence $\mathcal{K}$ is a compact (completely continuous) operator.

For the numerical solution of (1), we replace $\mathcal{K}$ by $\mathcal{K}_n$, and consider the approximating equation,

$$(\mathrm{I} - \mathcal{K}_n)\, f_n = F. \qquad \ldots \qquad (7)$$

Then, assuming the hypotheses,

(A) $\mathcal{K}_n f \to \mathcal{K} f$ for all $f \in c\,[0, 1]$

and

(B) the set $U = \{ \mathcal{K}_n f \mid n \geqslant 1 : \| f \| \leqslant 1 \}$

has compact closure in $c\,[0,1]$,

Anselone and Moore obtained the bound,

$$\| f - f_n \| \leqslant \| I - \mathcal{K}_n \|^{-1} \cdot \| \mathcal{K} F - \mathcal{K}_n F \| + \| K_n \mathcal{K} - \mathcal{K}^2 \| \; \| f \| \cdot \ldots \qquad (8)$$

Noting that,

$$\begin{aligned}(I-\mathscr{K}_n)(f-f_n) &= -F+(I-\mathscr{K}_n)f \\ &= -(I-\mathscr{K})f + (I-\mathscr{K}_n)f \\ &= (\mathscr{K}-\mathscr{K}_n)f = (\mathscr{K}-\mathscr{K}_n)\ (F+\mathscr{K}f) \\ &= \mathscr{K}F-\mathscr{K}_nF+(\mathscr{K}^2-\mathscr{K}_n\mathscr{K})\ f\end{aligned}$$

and assuming the existence of $(I-\mathscr{K}_n)^{-1}$, one obtains

$$f-f_n = (I-\mathscr{K}_n)^{-1}\ \{\mathscr{K}F-\mathscr{K}_nF + (\mathscr{K}^2-\mathscr{K}_n\mathscr{K})\ f\}$$

from which follows the result (8) above. In practice, however, the computation of $\| \mathscr{K}^2-\mathscr{K}_n\mathscr{K} \|$ is difficult, and a better bound is therefore necessary.

Assuming the split,

$$\mathscr{K}(s, \sigma) = H(s, \sigma)\ L\ (s, \sigma) \qquad \text{.. .. ..} \qquad (9)$$

where $L\ (s, \sigma)$ is continuous and $H(s,\sigma)$ is bounded, integrable and equicontinuous, Atkinson obtained the bound,

$$\| f-f_n \| \leqslant B\frac{h^2}{8}\ \|\mathscr{K}\|\ \ \frac{\partial^2 L(s,\sigma)\ f(\sigma)}{\partial\sigma^2}\ \|\ * \qquad (10)$$

for the generalized trapezoidal rule. Clearly, the order of convergence is h^2.

4. Numerical Example

In the study of boundary-layer development and in a variety of other applications, a study of the pressure distribution on a body of revolution is required. In general, the problem reduces to finding the solution of a linear integral equation. This equation is satisfied by the velocity distribution function on the body surface from which the pressure distribution can be found by Bernoulli's equation.

Let the generating curve of the body of revolution be parameterized by the distance along the curve measured from one of the two points where the curve meets the axis of revolution, $y=0$. If the total length of the curve be taken as 1, then any point on the curve may be written,

$$\left.\begin{aligned} x &= x(s), & y &= y(s) \\ \text{where}\quad x(0) &= 0 & y(0) &= y(1) = 0. \end{aligned}\right\} \qquad \text{.. .. ..} \qquad (11)$$

Imposing the conditions that $x' = \dfrac{dx}{ds}$, $y' = \dfrac{dy}{ds}$ are continuous and $x'^2+y'^2=1$, Vandrey (1951) deduced the following integral equation for the velocity distribution function $v(s)$:

$$v(s) = 2x'(s) - \frac{1}{\pi}\int_0^1 K(s,\sigma)\ v(\sigma)\ d\sigma,\ 0 \leqslant s \leqslant 1$$

$$K(s,\sigma) = \frac{1}{\sqrt{(x-\xi)^2+(y+\eta)^2}}\left[\frac{x'y-y'(x-\xi)}{y}K(k) - E(k)\left\{\frac{x'y-y'(x-\xi)}{y}+2\eta\frac{x'(y-\eta)-y'(x-\xi)}{(x-\xi)^2+(y-\eta)^2}\right\}\right] \quad \cdots \quad (12)$$

where

$$\left.\begin{aligned} \xi &= x(\sigma), \quad \eta=y(\sigma) \\ k^2 &= \frac{4y\eta}{(x-\xi)^2+(y+\eta)^2} \\ K(k) &= \int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \\ \text{and} \quad E(k) &= \int_0^{\pi/2}\sqrt{1-k^2\sin^2\theta}\,d\theta. \end{aligned}\right\} \quad (13)$$

Since the complete elliptic integral of the first kind $K(k)$ has a logarithmic singularity at $k^2=1$, the kernel $K(s,\sigma)$ will also have a logarithmic singularity when $\sigma=s$. Using the expansions of $K(k)$ and $E(k)$ given in Dwight (1934), the kernel $K(s,\sigma)$ can be split into the form:

$$K(s,\sigma) = P(s,\sigma)\log|s-\sigma| + Q(s,\sigma) \quad \cdots \quad \cdots \quad \cdots \quad (14)$$

where

$$\left.\begin{aligned} P(s,\sigma) &= \frac{1}{\sqrt{(x-\xi)^2+(y+\eta)^2}}\left[\frac{x'y-y'(x-\xi)}{y}\,\frac{2}{\pi}\,E(k_1) - 2\eta\frac{x'(y-\eta)-y'(x-\xi)}{(x-\xi)^2+(y-\eta)^2}\,\frac{2}{\pi}\left\{K(k_1)-E(k_1)\right\}\right] \\ Q(s,\sigma) &= K(s,\sigma)-P(s,\sigma)\log|s-\sigma| \end{aligned}\right\} \quad (15)$$

and $k_1^2 = 1-k^2$.

$$\left.\begin{aligned} &\text{When } \sigma = s, \text{ it is found that} \\ &P(s,s) = -\frac{x'}{2y} \\ &\text{and} \quad Q(s,s) = \frac{x'y''-y'x''}{2} - \frac{x'}{2y}(1-\log 8y). \end{aligned}\right\} \quad (16)$$

In the particular case of the body of revolution (sphere) considered here, at points where $y=0$, x' will be zero so that there $v(s)=0$, which are the stagnation points of the flow. For this equation, therefore, the discussion of § 2, holds, and the system of equations is:

$$\underset{\sim}{A}\,\underset{\sim}{v} = \underset{\sim}{B}$$

where

$$\left.\begin{aligned} a_{ij} &= P_{ij} \int_0^1 \left[p \log \left| \overline{i+1-j-ph} \right| + (1-p) \log \left| \overline{i-j-ph} \right| \right] dp \\ &\quad + Q_{ij}, \qquad \text{for} \quad i \neq j \\ &= P_{ij} \int_0^1 \left[p \log \left| \overline{i+1-j-ph} \right| + (1-p) \log \left| \overline{i-j-ph} \right| \right] dp \\ &\quad + Q_{ij} + \frac{\pi}{h}, \qquad \text{for} \quad i = j \\ &\qquad i = 1, 2, \ldots \ldots, n-1 \\ &\qquad j = 1, 2, \ldots \ldots, n-1, \\ \text{and} \quad b_i &= \frac{2\pi}{h} x_i', \; i = 1, 2, \ldots \ldots, n-1. \end{aligned}\right\} \quad (17)$$

A solution of (12) using Everett's formula was given by Kershaw (1961).

5. Discussion of Results

Numerical results for a sphere are given in the Table I. As the solution is centro-symmetric, the results are given only upto $s=90°$. The programme was run twice with $n=10$ and $n=20$. The accuracy is quite good, and the order of convergence is, clearly, h^2. This example shows that the generalized trapezoidal rule would generally be sufficient for practical problems.

Table I

s (in degrees)	Accurate value of $v(s)$ $= 1\cdot5 \sin s$	Computed $v(s)$	Error	Ratio
18	0·4635	$n = 10$ 0·4572	0·0063	4
		$n = 20$ 0·4619	0·0016	
36	0·8817	$n = 10$ 0·8812	0·0005	5
		$n = 20$ 0·8816	0·0001	
54	1·2135	$n = 10$ 1·2158	0·0023	4
		$n = 20$ 1·2141	0·0006	
72	1·4266	$n = 10$ 1·4303	0·0037	4
		$n = 20$ 1·4275	0·0009	
90	1·5	$n = 10$ 1·5042	0·0042	4
		$n = 20$ 1·5011	0·0011	

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