A NEW GEOMETRICAL APPROACH TO NON-SYMMETRIC UNIFIED FIELD THEORY

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The geometry of a pair of 'mutually exclusive' 4-dimensional varieties V_1 and V_2 embedded in a Minkowskian n-space is studied here by tensor methods. The two sets of tangent vectors $C_{,\mu}$ $D_{,\nu}$ at corresponding points of V_1 and V_2 define by their scalar products a non-symmetric $g_{\mu\nu}$ whose symmetric part $g_{\mu\nu}$ represents gravitational metric tensor $S_{\mu\nu}$ and the anti-symmetric part the electromagnetic tensor $F_{\mu\nu}$. But the connections here are symmetric in μ, ν. Starting from 'modified' sets of tangent vectors and applying tensor methods a geometry of unified field theory has been developed in which the connections are non-symmetric satisfying Einstein's equation but $S_{\mu\nu}$ and $F_{\mu\nu}$ are connected with the fundamental non-symmetric tensor of the field in a more complicated way.

In a recent paper (Ghosh 1970) it has been shown that the geometry of a four-fold can be studied in a more comprehensive way if it is considered embedded in a higher dimensional space. To accommodate all types of vectors having positive, negative or null vector-squares, as is required in relativity theory, the embedding space may be chosen Minkowskian. A Minkowskian n-space M_n is generated by a system of n mutually orthogonal coordinate vectors μ_i passing through an origin 0, of which the first m vectors have vector-squares each equal to 1 and the last $n-m$ vectors have vector-squares each equal to -1, so that the following scalar products have values given by

$$\begin{Bmatrix} \mu^i \\ \mu^j \end{Bmatrix} = 0, \; i \neq j = 1, 2, \ldots n,$$

$$\begin{Bmatrix} \mu^i \\ \mu^j \end{Bmatrix} = 1, \; i = j = 1, 2, \ldots m$$

$$\begin{Bmatrix} \mu^i \\ \mu^j \end{Bmatrix} = -1, \; i = j = m+1, \ldots n$$

Any vector or tensor of physical significance has some association with scalar products of vectors in M_n.

This embedding process may be interpreted kinematically as representing a generalized particle with n degrees of freedom which moves under constraint on a four-space, its position at any time being given by a set of four generalized coordinates. The $n-4$ equations of constraint will determine the four-space on which the particle moves.

§ 1. Immersed in the Minkowskian n-space defined above let us consider a pair of 4-dimensional varieties V_1 and V_2 represented by means of the equations

$$\left.\begin{aligned} \rho &= \sum_{i=1}^{n} S_i \mu^i \\ \rho &= \sum_{i=1}^{n} T_i \mu^i \end{aligned}\right\} \tag{1.1}$$

where S_i and T_i are 'mutually exclusive' analytic functions of four variables v^λ in a given region. Fixing upon a point P on V_1 and a point Q on V_2 corresponding to the same values of the parameters, the tangent subspaces at P and Q are then spanned by the two sets of M-vectors.

$$\begin{aligned} C,\ \lambda &= \sum_{i=1}^{n} \frac{\partial S_i}{\partial v^\lambda} \mu^i \, , \\ & \qquad\qquad (\lambda = 1, 2, 3, 4) \\ D,\ \lambda &= \sum_{i=1}^{n} \frac{\partial T_i}{\partial v^\lambda} \mu^i . \end{aligned} \tag{1.2}$$

Making use of the determinant g' symbolically represented by

$$g' = \begin{Bmatrix} C,_1 & C,_2 & C,_3 & C,_4 \\ D,_1 & D,_2 & D,_3 & D,_4 \end{Bmatrix} \tag{1.3}$$

which yields the (μ,ν)th element of g' as the scalar product of $C,_\mu\ D,_\nu$ one can obtain the contravariant vectors C'^μ, D'^ν associated to (1·2) expressed as

$$\left.\begin{aligned} C'^\mu &= \sum_{\lambda=1}^{4} C,_\lambda \begin{Bmatrix} C,_\lambda \\ D,_\mu \end{Bmatrix}_m \Big/ g' \\ D'^\nu &= \sum_{\lambda=1}^{4} D,_\lambda \begin{Bmatrix} D,_\lambda \\ C,_\nu \end{Bmatrix}_m \Big/ g' \end{aligned}\right\} \tag{1.4}$$

where the suffix m attached to an element of g' indicates that its minor with proper sign is to be taken. We now define the 'modified' tangent vectors C_μ, D_ν by means of the equations

$$\left.\begin{aligned} C_\mu &= C'^\lambda \begin{Bmatrix} D,_\lambda \\ C_\mu \end{Bmatrix} \\ D_\nu &= D'^\lambda \begin{Bmatrix} C,_\lambda \\ D_\nu \end{Bmatrix} \end{aligned}\right\} \tag{1.5}$$

the associated contravariant vectors being expressed as

$$\left.\begin{aligned} C^{\mu} &= C_{,\lambda} \begin{Bmatrix} D^{,\lambda} \\ C^{\mu} \end{Bmatrix} \\ D^{\nu} &= D_{,\lambda} \begin{Bmatrix} C^{,\lambda} \\ D^{\nu} \end{Bmatrix} . \end{aligned}\right\} \tag{1.6}$$

We now postulate

$$\left.\begin{aligned} g_{\mu\nu} &= \begin{Bmatrix} C_{\mu} \\ D_{\nu} \end{Bmatrix}, \quad g_{\nu\mu} = \begin{Bmatrix} C_{\nu} \\ D_{\mu} \end{Bmatrix} \\ g^{\mu\nu} &= \begin{Bmatrix} C^{\nu} \\ D^{\mu} \end{Bmatrix}; \quad g^{\nu\mu} = \begin{Bmatrix} C^{\mu} \\ D^{\nu} \end{Bmatrix} . \end{aligned}\right\} \tag{1.7}$$

Differentiating C_{μ}, D_{ν} one can verify

$$\left.\begin{aligned} C_{\mu\nu} &= \partial_{\nu} C_{\mu} \mp C_{\nu\mu} \\ D_{\nu\mu} &= \partial_{\mu} D_{\nu} \mp D_{\mu\nu} . \end{aligned}\right\} \tag{1.8}$$

Let us now consider the transformation formulae

$$\left.\begin{aligned} C'_{\mu_1 \mu_2} &= C_{\lambda_1\lambda_2} \frac{\partial v^{\lambda_1}}{\partial v'^{\mu_1}} \frac{\partial v^{\lambda_2}}{\partial v'^{\mu_2}} + C_{\lambda} \frac{\partial^2 v^{\lambda}}{\partial v'^{\mu_1} \partial v'^{\mu_2}} \\ D'_{\mu_1 \mu_2} &= D_{\lambda_1\lambda_2} \frac{\partial v^{\lambda_1}}{\partial v'^{\mu_1}} \frac{\partial v^{\lambda_2}}{\partial v'^{\mu_2}} + D_{\lambda} \frac{\delta^2 v^{\lambda}}{\delta v'^{\mu_1} \delta v'^{\mu_2}} \end{aligned}\right\} \tag{1.9}$$

subject to the general transformation scheme

$$v'^{\lambda} = \phi^{\lambda}(v^1, v^2, v^3, v^4) . \tag{1.10}$$

It can be shown easily that each of the M-vectors

$$\left.\begin{aligned} C^{*}_{\lambda_1\lambda_2} &= C_{\lambda_1\lambda_2} - C_{\lambda} \begin{Bmatrix} D^{\lambda} \\ C_{\lambda_1 \lambda_2} \end{Bmatrix}, \quad C^{\dagger}_{\lambda_1\lambda_2} = C_{\lambda_1\lambda_2} - C_{\lambda} \begin{Bmatrix} C^{\lambda} \\ D_{\lambda_1\lambda_2} \end{Bmatrix} \\ D^{*}_{\lambda_1\lambda_2} &= D_{\lambda_1\lambda_2} - D_{\lambda} \begin{Bmatrix} C^{\lambda} \\ D_{\lambda_1\lambda_2} \end{Bmatrix}, \quad D^{\dagger}_{\lambda_1\lambda_2} = D_{\lambda_1\lambda_2} - D_{\lambda} \begin{Bmatrix} D_{\lambda} \\ C_{\lambda_1\lambda_2} \end{Bmatrix} \end{aligned}\right\} \tag{1.11}$$

is a covariant tensor of the second rank in the v-space. From (1·11) one obtains the relations

$$\begin{Bmatrix} C^{*}_{\lambda_1 \lambda_2} \\ D^{\lambda} \end{Bmatrix} = \begin{Bmatrix} D^{*}_{\lambda_1 \lambda_2} \\ C^{\lambda} \end{Bmatrix} = 0, \begin{Bmatrix} C^{\dagger}_{\lambda_1 \lambda_2} \\ D^{\lambda} \end{Bmatrix} = - \begin{Bmatrix} D^{\dagger}_{\lambda_1 \lambda_2} \\ C^{\lambda} \end{Bmatrix} = \begin{Bmatrix} C_{\lambda_1 \lambda_2} \\ D^{\lambda} \end{Bmatrix} - \begin{Bmatrix} D_{\lambda_1 \lambda_2} \\ C^{\lambda} \end{Bmatrix} . \tag{1·12}$$

We now postulate

$$\begin{Bmatrix} D^{\lambda} \\ C_{\lambda_1 \lambda_2} \end{Bmatrix} = \begin{Bmatrix} C^{\lambda} \\ D_{\lambda_2\lambda_1} \end{Bmatrix} = \Gamma^{\lambda}_{\lambda_1 \lambda_2}, \tag{1·13}$$

imposing some restrictions on the functions S_i, T_i defining the varieties V_1 and V_2.

The identical relation

$$\delta_\sigma \begin{Bmatrix} C_{\lambda_1} \\ D_{\lambda_2} \end{Bmatrix} = \begin{Bmatrix} C_{\lambda} \\ D_{\lambda_2} \end{Bmatrix} \begin{Bmatrix} C_{\lambda_1\sigma} \\ D^{\lambda} \end{Bmatrix} + \begin{Bmatrix} D_{\lambda} \\ C_{\lambda_1} \end{Bmatrix} \begin{Bmatrix} C^{\lambda} \\ D_{\lambda_2\sigma} \end{Bmatrix} \tag{1·14}$$

then becomes the non-symmetric equations of connection

$$\delta_\sigma g_{\lambda_1\lambda_2} = g_{\lambda\lambda_2} \Gamma^{\lambda}{}_{\lambda_1\sigma} + g_{\lambda_1\lambda} \Gamma^{\lambda}{}_{\sigma\lambda_2} \tag{1.15}$$

in Einstein's unified field theory.

§ 2. We now proceed to find appropriate representations for the gravitational metric tensor and the electromagnetic tensor in the present formulation.

Consider the identical relation

$$\delta_\sigma \begin{Bmatrix} C_{,\mu} \\ D_{,\nu} \end{Bmatrix} = \begin{Bmatrix} D^{,\lambda} \\ C_{,\mu\sigma} \end{Bmatrix} \begin{Bmatrix} C_{,\lambda} \\ D_{,\nu} \end{Bmatrix} + \begin{Bmatrix} C^{,\lambda} \\ D_{,\nu\sigma} \end{Bmatrix} \begin{Bmatrix} D_{,\lambda} \\ C_{,\mu} \end{Bmatrix}. \tag{2.1}$$

Combining with its conjugate (C, D interchanged) we obtain

$$\left.\begin{aligned} \delta_\sigma S_{\mu\nu} &= S_{\lambda\nu} P^{\lambda}_{\mu\sigma} - F_{\lambda\nu} V^{\lambda}_{\mu\sigma} + S_{\lambda\mu} P^{\lambda}_{\nu\sigma} - F_{\lambda\mu} V^{\lambda}_{\nu\sigma} \\ \delta_\sigma F_{\mu\nu} &= - S_{\lambda\nu} V^{\lambda}_{\mu\sigma} + F_{\lambda\nu} P^{\lambda}_{\mu\sigma} + S_{\lambda\mu} V^{\lambda}_{\nu\sigma} - F_{\lambda\mu} P^{\lambda}_{\nu\sigma} \end{aligned}\right\} \tag{2.2}$$

where

$$\begin{aligned} S_{\mu\nu} &= \tfrac{1}{2}\left(\begin{Bmatrix} C_{,\mu} \\ D_{,\nu} \end{Bmatrix} + \begin{Bmatrix} D_{,\mu} \\ C_{,\nu} \end{Bmatrix} \right) \\ F_{\mu\nu} &= \tfrac{1}{2}\left(\begin{Bmatrix} C_{,\mu} \\ D_{,\nu} \end{Bmatrix} - \begin{Bmatrix} D_{,\mu} \\ C_{,\nu} \end{Bmatrix} \right) \\ P^{\lambda}_{\mu\nu} &= \tfrac{1}{2}\left(\begin{Bmatrix} C^{,\lambda} \\ D_{,\mu\nu} \end{Bmatrix} + \begin{Bmatrix} D^{,\lambda} \\ C_{,\mu\nu} \end{Bmatrix} \right) \\ V^{\lambda}_{\mu\nu} &= \tfrac{1}{2}\left(\begin{Bmatrix} C^{,\lambda} \\ D_{,\mu\nu} \end{Bmatrix} - \begin{Bmatrix} D^{,\lambda} \\ C_{,\mu\nu} \end{Bmatrix} \right). \end{aligned} \tag{2·3}$$

From (2·2) one can show

$$S^{\rho}{}_{\mu\nu} = P^{\rho}{}_{\mu\nu} - S^{\rho\sigma} F_{\lambda\sigma} V^{\lambda}{}_{\mu\nu} \tag{2·4}$$

where

$$S^{\rho}{}_{\mu\nu} = \tfrac{1}{2} S^{\rho\sigma} (\delta_\mu S_{\nu\sigma} + \delta_\nu S_{\sigma\mu} - \delta_\sigma S_{\mu\nu}). \tag{2·5}$$

Further, we obtain

$$\left.\begin{aligned} \delta_\sigma F_{\mu\nu} + \delta_\mu F_{\nu\sigma} + \delta_\nu F_{\sigma\mu} = 0 \\ \delta_\sigma S_{\mu\nu} - S_{\lambda\nu} S^{\lambda}{}_{\mu\sigma} - S_{\lambda\mu} S^{\lambda}{}_{\nu\sigma} = 0 \end{aligned}\right\}. \tag{2.6}$$

We identify $F_{\mu\nu}$ with the electromagnetic tensor and $S_{\mu\nu}$ with the gravitational metric tensor.

Let us now construct the Riemann Christoffel tensor. In (1·11) we observe that the covariant differentiation of C_λ and D_λ occurs in two forms. For the contravariant vector C^λ the two forms of covariant differentiation are expressed as

$$(C^\lambda)^*_{;\nu} = \delta_\nu C^\lambda + C^\mu \begin{Bmatrix} D^\lambda \\ C_{\mu\nu} \end{Bmatrix} \tag{2.7}$$

$$(C^\lambda)^\dagger_{;\nu} = \delta_\nu C^\lambda + C^\mu \begin{Bmatrix} C^\lambda \\ D_{\mu\nu} \end{Bmatrix}.$$

Similar are the two forms of covariant differentiation for D^λ.

Let us now consider the two scalar products

$$\begin{Bmatrix} (D^\lambda)^*{}_{;\nu} \\ C^*{}_{\lambda_1\lambda_2} \end{Bmatrix} = \begin{Bmatrix} \delta_\nu D^\lambda \\ C_{\lambda_1\lambda_2} \end{Bmatrix} + \begin{Bmatrix} D^\mu \\ C_{\lambda_1\lambda_2} \end{Bmatrix} \begin{Bmatrix} D^\lambda \\ C_{\mu\nu} \end{Bmatrix} \tag{2.8}$$

$$\begin{Bmatrix} (C^\lambda)^\dagger_{;\nu} \\ D^*{}_{\lambda_1\lambda_2} \end{Bmatrix} = \begin{Bmatrix} \delta_\nu C^\lambda \\ D_{\lambda_1\lambda_2} \end{Bmatrix} + \begin{Bmatrix} C^\mu \\ D_{\lambda_1\lambda_2} \end{Bmatrix} \begin{Bmatrix} C^\lambda \\ D_{\mu\nu} \end{Bmatrix}. \tag{2.9}$$

In (2·8) interchanging ν and λ_2 and subtracting one from the other we obtain the Riemann-Christoffel tensor

$$R^{\lambda}{}_{\lambda_1\lambda_2\nu} = \delta_\nu \begin{Bmatrix} D^\lambda \\ C_{\lambda_1\lambda_2} \end{Bmatrix} - \delta_{\lambda_2} \begin{Bmatrix} D^\lambda \\ C_{\lambda_1\nu} \end{Bmatrix}$$
$$+ \begin{Bmatrix} D^\mu \\ C_{\lambda_1\lambda_2} \end{Bmatrix} \begin{Bmatrix} D^\lambda \\ C_{\mu\nu} \end{Bmatrix} - \begin{Bmatrix} D^\mu \\ C_{\lambda_1\nu} \end{Bmatrix} \begin{Bmatrix} D^\lambda \\ C_{\mu\lambda_2} \end{Bmatrix} \tag{2.10}$$

By a similar procedure from (2·9) we derive its conjugate form (C, D interchanged)

$$\tilde{R}^{\lambda}{}_{\lambda_1\lambda_2\nu}.$$

In the present formulation of Einstein's unified field theory the set of equations

$$\begin{Bmatrix} C_{;\mu} \\ D_{;\nu} \end{Bmatrix} = \begin{Bmatrix} C^\lambda \\ D^\rho \end{Bmatrix} \begin{Bmatrix} D_\lambda \\ C_{;\mu} \end{Bmatrix} \begin{Bmatrix} C_\rho \\ D_{;\nu} \end{Bmatrix}$$
$$\begin{Bmatrix} D_{;\mu} \\ C_{;\nu} \end{Bmatrix} = \begin{Bmatrix} D^\lambda \\ C^\rho \end{Bmatrix} \begin{Bmatrix} C_\lambda \\ D_{;\mu} \end{Bmatrix} \begin{pmatrix} D_\rho \\ C_\nu \end{pmatrix} \tag{2.11}$$

shows that the electromagnetic tensor $F_{\mu\nu}$ and the gravitational metric tensor $S_{\mu\nu}$ are linearly connected with $g^{\lambda\rho}_{-}$ and $g^{\lambda\rho}_{\nu}$. In Einstein's formulation no such relation comes in.

The connection coefficients $\left\{ \begin{matrix} D_\lambda \\ C_{,\mu} \end{matrix} \right\}$ and $\left(\begin{matrix} C_\lambda \\ D_{,\mu} \end{matrix} \right)$ involved in (2·11) are obtained by solving algebraically the following two sets of equations :—

$$R^{\lambda}_{\ \rho\sigma\nu} \left(\begin{matrix} C_\lambda \\ D_{,\mu} \end{matrix} \right) + R^{\lambda}_{\ \rho\nu\mu} \left(\begin{matrix} C_\lambda \\ D_{,\sigma} \end{matrix} \right) + R^{\lambda}_{\ \rho\mu\sigma} \left(\begin{matrix} C_\lambda \\ D_{,\nu} \end{matrix} \right) = 0,$$

$$\tilde{R}^{\lambda}_{\ \rho\sigma\nu} \left(\begin{matrix} D_\lambda \\ C_{,\mu} \end{matrix} \right) + \tilde{R}^{\lambda}_{\ \rho\nu\mu} \left(\begin{matrix} D_\lambda \\ C_{,\sigma} \end{matrix} \right) + \tilde{R}^{\lambda}_{\ \rho\mu\sigma} \left(\begin{matrix} D_\lambda \\ C_{,\nu} \end{matrix} \right) = 0.$$

The actual determination of these coefficients is, however, a complicated process. It may be noted from (2·6) that if $g_{\mu\nu}$ is defined by $\left(\begin{matrix} C_{,\mu} \\ D_{,\nu} \end{matrix} \right)$ then $g_{\underline{\mu\nu}}$ is $S_{\mu\nu}$ and $g_{\underset{\vee}{\mu\nu}}$ is $F_{\mu\nu}$.

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Reference

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